# Parametric Resonance and Elastic Pendulums

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**Abstract**

In this I try to extend the theoretical conception of Elastic Pendulum that can be explained by the Driven Pendulums that I presented during my Classical Mechanics course, to explain the concept of *Parametric Resonance* and devise an experiment for the same. The actual experiment struck me when I saw the elastic pendulum in the lab.

## Aim

As stated in the abstract we use a Elastic Pendulum in realizing the concept of Parametric Resonance. This deals with the conception of Parametric Resonance and an experimental setup to verify these.

## 1 Theory

### Definition

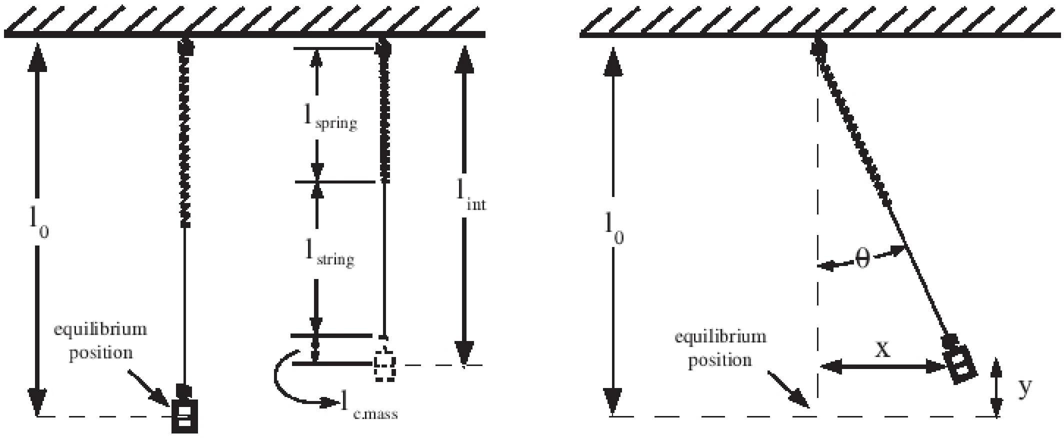
Parametric Resonance is the parametrical resonance phenomenon of mechanical excitation and oscillation at certain frequencies and their associated harmonics. The main difference between this and the regular resonance is that parametric resonance is an instability phenomenon.

### A brief introduction to Elastic Pendulums

Elastic Pendulums comprise of a mass hanging from a rigid support with the help of a spring. This is a highly non-linear system in the sense that the motion is highly sensitive to the initial conditions when compared to linear systems. In most of the cases, the mass of the spring is neglected as it is much smaller than the mass of the weight hanging from it.

Actually, the Elastic Pendulum is a very simple system. But, it provides a rich variety of non-linear phenomenon. For some values of parameters and initial conditions, the motion can be chaotic too. To describe heuristically, the elastic pendulum has two different kinds of motion called “lemon” and “crescent”. The lemon motion gets it’s name from the fact that the motion traces out a lemon shape with a slice taken out either from top or bottom depending on the initial conditions i.e. the motion is bounded by three parabolas; two determining the outer boundary and one determining the nature of slice.

Let's look at the mathematics behind the elastic pendulum. From the figure below we can see that we set up a coordinate system with origin at the equilibrium position.




The equilibrium length is denoted as $l_0$. If $T$ is the tension pointing inward, towards the pivot hook of the spring and $mg$ is the force of gravity on the weight, then,

$$\sum F_x = m\ddot{x}(t) = -T\sin\theta$$

$$\sum F_y = m\ddot{y}(t) = T\cos\theta - mg$$

We can use the Hooke's law for the spring and determine the spring force as $F = -kz$ where $k$ is the *spring constant*. If $l_{int}$ is the initial length of the spring without the weight attached, then $l(t) = \sqrt{x(t)^2 + [l_0 - y(t)]^2}$ is the length of the spring with the weight attached as a function of time. Hence,

$$T = k[l(t) - l_{int}$$

is the Tension in the Spring. At equilibrium position when $l(t) = l_0$, $k[l_0 - l_{int}] = mg$. From this and from the force equations we can get that

$$m\ddot{x}(t) = \frac{k[l(t) - l_{int}]x(t)}{l(t)}$$

$$m\ddot{y}(t) = \frac{k[l(t) - l_{int}][l_0 - y(t)]}{l(t)} - k[l_0 - l_{int}]$$

where $sin\theta = x(t)/y(t)$ and hence $cos\theta = [l_0 - y(t)]/l(t)$.

Now, to solve for $x(t)$ and $y(t)$, $l(t)$ must be written in terms of these. But this will make the equation cluttered. So, let's try to approximate using the Taylor's expansion to get

$$m\ddot{x}(t) \approx -k\left\{1 - \frac{l_{int}}{l_0}\right\}x(t) + k\frac{l_{int}}{l_0^2}x(t)y(t)$$

$$m\ddot{y}(t) \approx -ky(t) + \frac{kl_{int}}{2l_0^2}x(t)^2$$

Substituting, $\lambda = \frac{kl_{int}}{ml_0^2}$, we can rewite as

$$\ddot{x}(t) + \omega_p^2 x(t) \approx \lambda x(t)y(t)$$

$$\ddot{y}(t) + \omega_x^2 y(t) \approx \frac{\lambda}{2}x(t)^2$$

where

$$\omega_s^2 = \frac{k}{m}$$

$$\omega_p^2 = \frac{k}{m}\left\{1 - \frac{l_{int}}{l_0}\right\} = \frac{g}{l_0}$$

The quantity $\omega_p$ is the angular frequency at which a regular pendulum oscillates, while $\omega_s$ is the angular frequency at which a one-dimensional mass on a spring($i.e.$ the elastic pendulum allowed to move only in vertical direction) oscillates. The significance of this is that, if the terms on the right hand side of the above equations were zero, the motion in the $y$ direction would just be that of a mass on a spring in one dimension; while motion in the $x$ direction would be that of a regular pendulum, and the two would be independent of one another. This means the terms on the right hand side of the equations constitute nonlinear coupling terms, binding the motion of each direction to the other. These equations are coupled, meaning one cannot be solved without solving the other simultaneously.

## Stability analysis and the entry of Parametric Resonance

To understand the motion, first let's take a look at the situation where the spring motion dominates the pendulum motion, $x(t) \ll y(t)$, in the beginning. Then we get, (after setting $\lambda x(t)^2/2$ term to zero)

$$y(t) = a\cos(\omega_s t)$$

where $a$ is a constant. From this we get

$$\ddot{x}(t) + [\omega_p^2 - \lambda a\cos(\omega_s t)]x(t) = 0$$

This type of differential equation is called as *Mathieu Equation*. According, to this , when

$$\frac{\omega_s}{\omega_p} = \frac{2}{n}$$

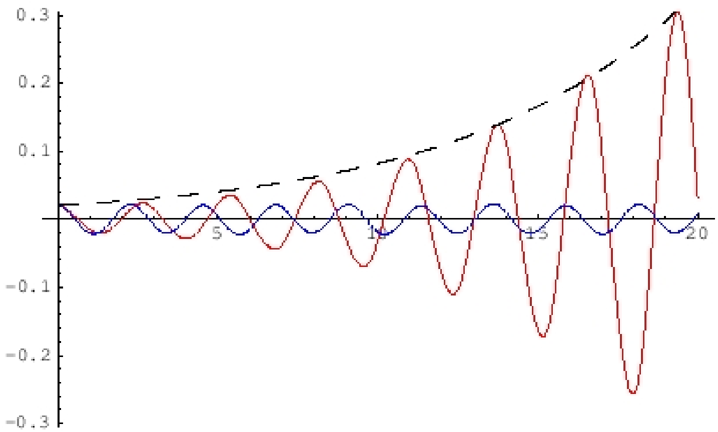
where $n$ is a positive integer, the solutions are unstable; otherwise the solutions are stable. Instability is most pronounced when $n = 1$. From this we can derive a relation for $l_0$ and $l_{int}$ at $n = 1$ for which the system exhibits instability.

$$l_0 = \frac{4}{4 - n^2} l_{int} \Rightarrow l_0 = \frac{4}{3} l_{int}$$

The physical significance of finding an instability when $\omega_s/\omega_p = 2$ is that for any initial condition $x_{int}$ , of the elastic pendulum that is not exactly zero, $x(t)$ will oscillate with an *exponentially increasing amplitude.*

According to this equation, even if it was started with initial conditions $x_{int} = 0.025m$ and $y_{int} = 0.25m$, making the amplitude of oscillations in the $y$ direction 10 times larger to begin with, the amplitude of $x(t)$ would eventually grow and overtake $y(t)$. Of course, this exponential growth cannot go on forever; the approximation that leads to the above equation of motion, that the motion in the $x$ direction was much smaller than motion in the $y$ direction, breaks down eventually, but it is surprising how well the above approximation models the initial motion of $x(t)$.

This is the rise of Parametric Resonance. Parametric resonance in general refers to instability which is sensitive to parameters of the system, which in this case are the *mass of the weight*, $l_{int}$, $l_0$, $k$ and the gravitational acceleration constant $g$. If $\omega_s/\omega_p = 1.5 or 2$, for example, the amplitude of $x(t)$ when the elastic pendulum was started at $x_{int} = 0.025m$ and $y_{int} = 0.25m$ would grow only a tiny bit. This solution is stable; a small deviation in the initial conditions ($x_{int} = 0.025m$ instead of exactly zero) leads to a small deviation in output, meaning the elastic pendulum would oscillate with a small amplitude around zero in the $x$ direction. We can see this from the plot below.




## 2 Possible Experimental Setup

As this is non-linear system, the setup should be constructed very precisely. For the setup, we need a spring of known spring constant (if not known this can be calculated), a mass, a rigid support stand, a stand of board with a graph paper attached to it, a web camera with a computer. All these are available in the lab, and over these we might have to do a little bit of coding exercises video manipulation and then plotting which can be done pretty easily.

First, we take all the necessary measurements, like the mass of weight, lengths of the spring, etc. Now, hang the pendulum and then attach the mass which is the usual method. For the sake of alignment, we can use a short stretch (of known length) of string, so that when we put the graph sheet on the board we can match the y-axis with the string and the center of mass of weight to the center of grid. The camera also should be aligned such that the center coincides with the center of the grid sheet. After all this tedious alignment, which seems pretty difficult to be done. We can start our experiment.

As our equations of motion are in a plane, we should make sure that there are no impetus to destroy this, so that our life will not be more complicated. But, this is totally dependent on us and cannot be set automatically, so we need to be careful with the air movement in the room and also the initial conditions that we give to the pendulum. And, one more thing is that, as we are using a spring, there can be rotations, which we have neglected. This should also be minimized by taking readings only when the system is in rotational equilibrium.

For measuring $y_{int}$, we should lift the spring from it's equilibrium position to the point where it is no longer stretched. The $y_{int}$ should be little less than the stretched length of the spring in the negative $y$ direction. The other initial condition, $x_{int}$ should be a small value, probably a couple of centimeters, because, if the $x_{int}$ is a little off from zero, the distance the mass swings from the $y - axis$ increases exponentially. Depending, on the are spanned for this, we have to position our camera such the complete proceedings are recorded with the graph sheet well resolved. We can show the parametric nature by choosing different masses or lengths of strings, such that $\omega_s/omega_p \neq 2$.

From the above recordings, we get a set of data. We also solve the Mathieu Equation numerically using a C or a Matlab code and use this to match with the data from the experiment. We cannot expect the experimental data to match the numerical one as we neglected air-drag. So, we might have a short time from the start of motion where it matches the numerical solution.