An "Electric" Story of a Drunkard

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Outline

The Problem Statement

Random Walk: You are placed at some point in a $d-$dimensional integer lattice.

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Repeat {
Pick one of your neighbours uniformly at random
and move to them
}
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Random Walk: You are placed at some point in a $d-$dimensional integer lattice.

Repeat {

`Pick one of your neighbours uniformly at random and move to them`

}

Question: What is the probability that you return to the starting point infinitely often?

The Theorem of Polya

Theorem

A random walk on an d-dimensional integer lattice will return to the starting point infinitely often with probability 1 *if* $d = 1$ *or* 2. *The probability is* 0 *if* $d \geq 3$.

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Drunkard: Will I ever get home again?
Polya: You can't miss, just keep walking, and stay out of $3D$!!!

Structure of this Talk

- Establish certain electrical networks

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- Estimate the effective resistance of the electrical networks
- Show the relation between electrical parameters of the networks and random walks
- Use the effective resistance of networks to prove Polya's theorem.

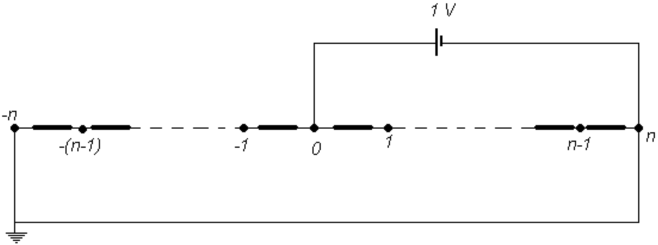
Ohm...Kirchoffāya Namahā

Ohm's Law: The current flowing through a wire is directly proportional to potential across it.

Kirchoff's Law: Current in = current out, for any point in the interior of the circuit.
Net potential drop across a loop is zero

Outline

Circuit 1



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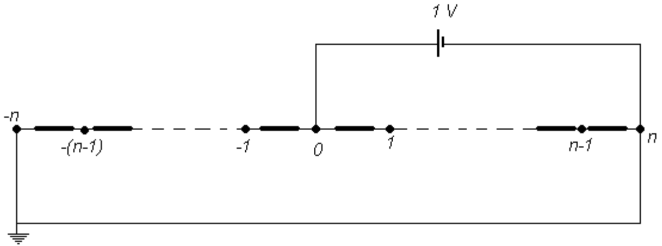

Effective Resistance

$$\rho_n = \frac{rn}{2}$$

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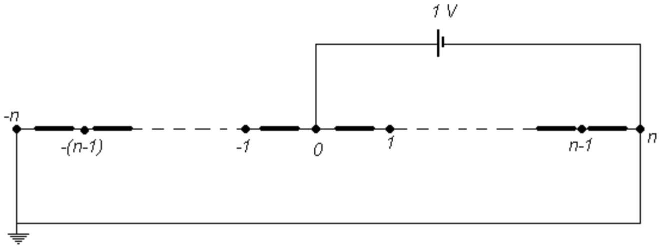
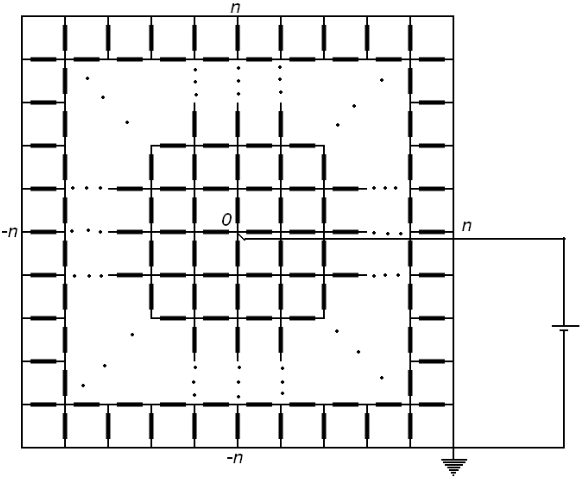


Effective Resistance

$$\rho_n = \frac{rn}{2}$$

$$\rho_n \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

Circuit 2



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Difficult to find a closed form answer

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But what happens to ρ_n as $n \to \infty$?

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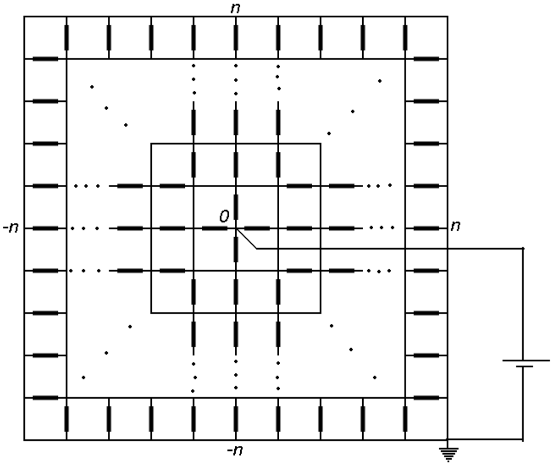
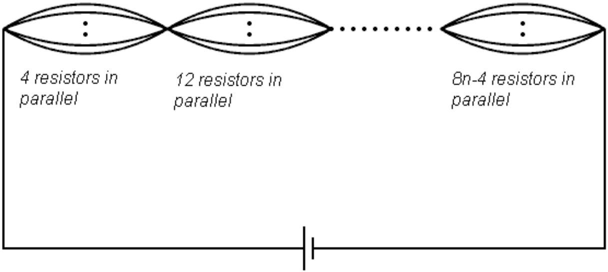
But what happens to ρ_n as $n \to \infty$?

Find an easier circuit with effective resistance $\rho' < \rho$ and see if you can show that ρ' is infinite.

An Easier Circuit



...which is equivalent to



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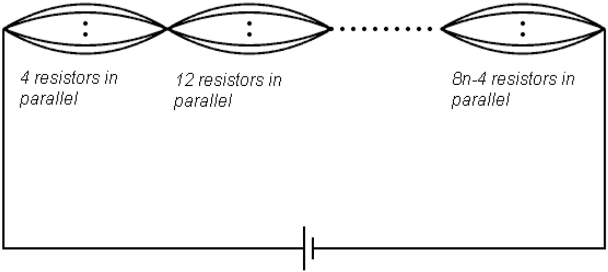

$$\rho'_n = \sum_{k=1}^{n} \frac{r}{8k-4}$$

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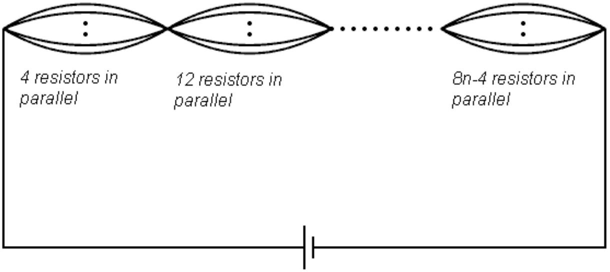

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Hence, we have $\rho'_n \to \infty$ as $n \to \infty$

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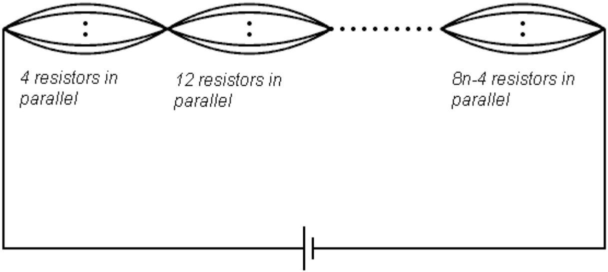

$$\rho'_n = \sum_{k=1}^{n} \frac{r}{8k-4}$$

Hence, we have $\rho'_n \rightarrow \infty$ as $n \rightarrow \infty$, which implies that $\rho_n \rightarrow \infty$ as $n \rightarrow \infty$.

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What can you say about ρ_n as $n \to \infty$?

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Again, difficult to find a closed form solution

What can you say about ρ_n as $n \longrightarrow \infty$?

Can you show that it is finite?

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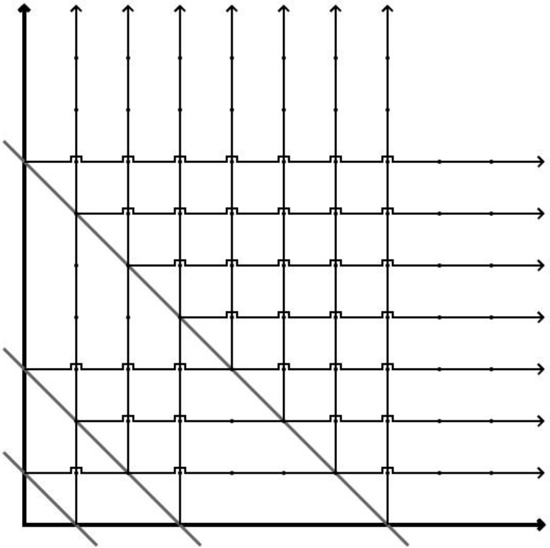
Show that the effective resistance of the new circuit is bounded.

Getting Back to 2D



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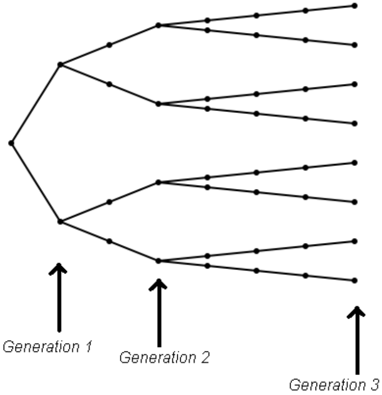

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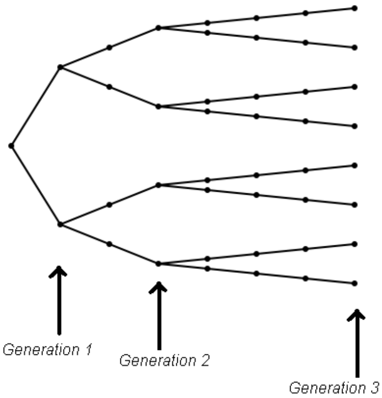
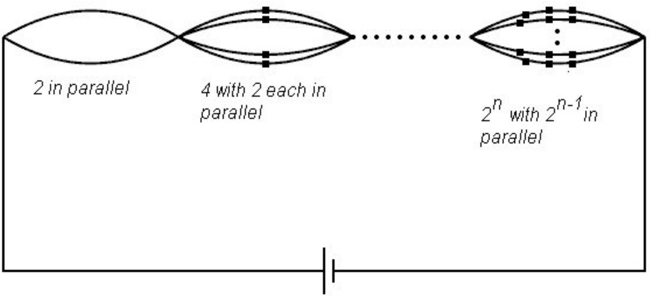


All *Generation i* nodes are of the same potential; can connect them without changing effective resistance.

...which is equivalent to



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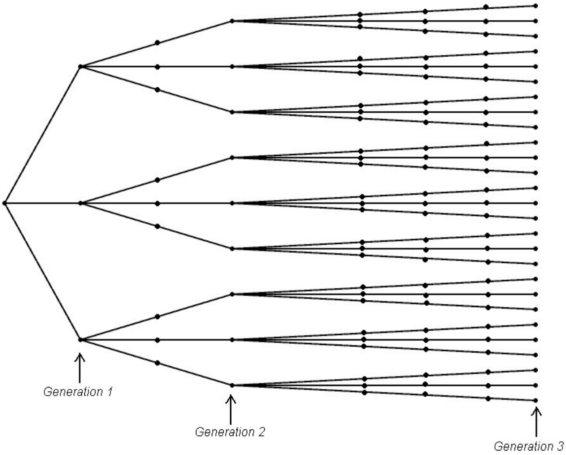
- Establish the tree again

The Tree



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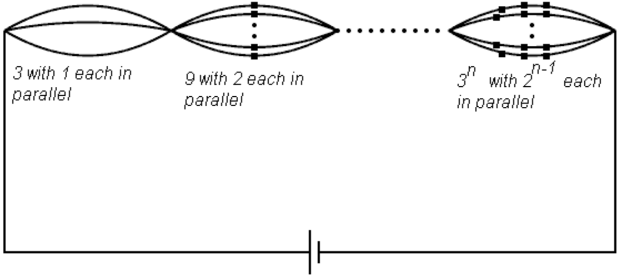

Effective resistance

$$\rho'_n = \frac{r}{3} + \frac{2r}{9} + \ldots + \frac{2^{n-1}r}{3^n}$$
$$= \left(1 - \left(\frac{2}{3}\right)^n\right) r$$

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Hence ρ for circuit 3 is finite.

Circuit d: $d > 3$

- Use the same approach, draw the corresponding planes
- Branch whenever it the lines hit the plane

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Finite for $d = 3 \Rightarrow$ finite for $d > 3$

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- Let G be a graph, with two special nodes α, β. Suppose we start at some node x, what is the probability $p(x)$ that a simple random walk starting from x reaches α before β.

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Claim: Both are the same!

Some Notations

R_{xy}	:	Resistance of the edge $\{x, y\}$
C_{xy}	:	Conductance of the edge, defined as $\frac{1}{R_{xy}}$
C_x	:	Conductance of the vertex, defined as $\sum_{y \in \Gamma(x)} C_{xy}$
i_{xy}	:	Current through the edge $\{x, y\}$
$v(x)$	:	Potential at the point x
i_α	:	Net current flowing through the circuit

Some Equations

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And of course, $v(\alpha) = 1, v(\beta) = 0$.

Defining Equations for $p(x)$

$$P(x) = \begin{cases} \frac{1}{\#\text{ of vertices adjacent to } x} & \text{if } y \text{ adjacent to } x \\ 0 & \text{otherwise} \end{cases}$$

... the probability that you reach y from x in 1 step.

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Hey, this looks familiar...

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Now we have:

$$p(x) = \sum_{y \in \Gamma(x)} p(y) P_{xy}; p(\alpha) = 1; p(\beta) = 1$$

Hey, this looks familiar...

$$v(x) = \sum_{y \in \Gamma(x)} v(y) \frac{C_{xy}}{C_x}; v(\alpha) = 1; v(\beta) = 0$$

Bingo!

If,

$$P_{xy} = \frac{C_{xy}}{C_x}$$

then we have

$$p(x) = v(x)$$

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If,

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then we have

$$p(x) = v(x)$$

...connection established!

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Thus we are hunting for

$$\xi = \sum_{y \in \Gamma(\alpha)} p(y) P_{\alpha y}$$

Net Current

$$i_\alpha = \sum_{y \in \Gamma(x)} \left(v(\alpha) - v(y)\right) C_{\alpha y}$$

Net Current

$$\begin{aligned} i_\alpha &= \sum_{y \in \Gamma(x)} (v(\alpha) - v(y))\, C_{\alpha y} \\ &= v(\alpha) \sum_{y \in \Gamma(x)} C_{\alpha y} - \sum_{y \in \Gamma(x)} v(y) C_{\alpha y} \end{aligned}$$

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\end{aligned}
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\therefore \sum_{y \in \Gamma(x)} p(y) P_{xy} = \xi = 1 - \frac{i_\alpha}{C_\alpha} = 1 - \frac{1}{\rho C_\alpha}
$$

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For $d = 1$ or 2, we have $\rho = \infty$ and hence $\xi = 1$.

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□

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End Notes

In probability jargon, it is said that a simple random walk is recurrent in 2 or lower dimensions, and is transient in 3 or higher dimensions.

The original question was asked and solved by George Polya in a celebrated paper in 1921. This method of looking at the question through electrical networks is based on a monograph by Doyle and Snell, where besides the question of transience and recurrence, many other questions of simple random walks are addressed.

Thank You

Slides and TeX files are available at
~ramprasad/studenttalks/drunkard/.