The Banach-Tarski Paradox

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The Statement: Informal

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One can pick a representative of each box

The Statement: Formal

Axiom of Choice

Let $\mathcal{A}$ be a collection of non-empty sets. Then, there exists a *choice function*

$$f : \mathcal{A} \longrightarrow \bigcup_{A \in \mathcal{A}} A$$

such that $f(A) \in A$ for all $A \in \mathcal{A}$.

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- Required for various fundamental results:
    - Every vector space has a basis
    - Every field has an algebraic closure
    - Given two sets A, B, either $|A| > |B|$ or $|B| > |A|$ or $|A| = |B|$
    - Product of non-empty sets is non-empty

What’s the harm in using it?

Well… some consequences of AoC are extremely counter-intuitive, like the *well-ordering theorem*.

What’s the harm in using it?

Well... some consequences of AoC are extremely counter-intuitive, like the *well-ordering theorem*. The most bizarre (IMHO) being the Banach-Tarski Paradox.

The Ultimate Jigsaw Puzzle

Proposition (Banach-Tarski)

Assuming the axiom of choice, it is possible to partition the solid unit sphere into 6 *pieces* $A_1,\ldots,A_6$*, and just rotate/translate these pieces to* $A_1',\ldots,A_6'$ *which a partition of two disjoint unit spheres.*

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Disclaimer: I am being a mathematician. By sphere, I mean

$$\mathcal{B}_3 = \left\{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 \leq 1\right\}$$

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- We'll see a proof, not involving 6 but certainly less than 20 pieces.

The Circle and the Broken Circle

$$
\begin{aligned}
\mathcal{S}_1 &= \{z \in \mathbb{C} \ : \ |z| = 1\} \\
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$\rho : z \mapsto e^{i} z$, rotation by one radian counterclockwise.

$A' = \rho(A)$

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Keep B as it is and we get the broken circle.

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Appetiser to the Proof

Broken Circle ≡ Circle

$F_2 \equiv F_2 \cup F_2$

Doubling the Sphere

Preliminaries

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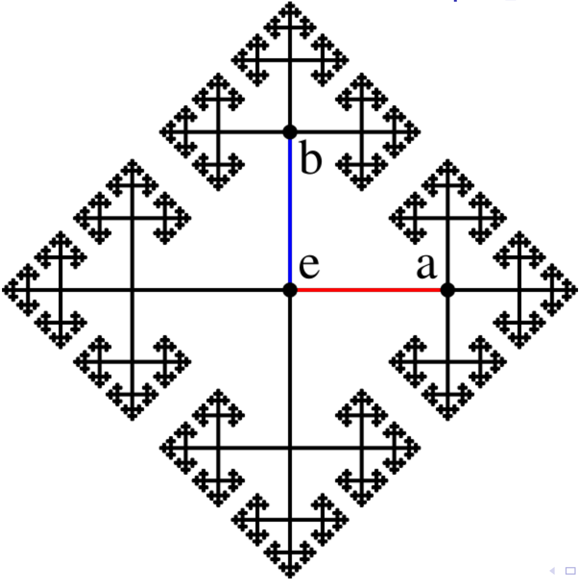
Conclusion

The Free Group

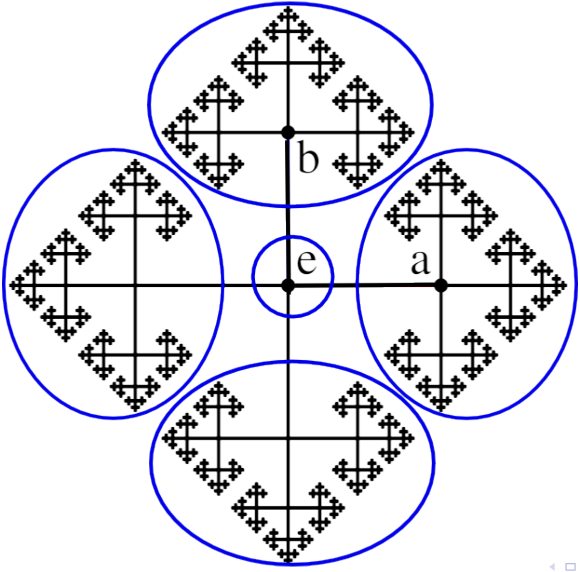
The Free Group over 2 Generators

The elements of this groups are strings over a, a^{-1}, b, b^{-1} where multiplication is concatenation. And you collapse sequences like aa^{-1} to the identity of the empty string. Hence, $aabb^{-1}a^{-1} = a$.

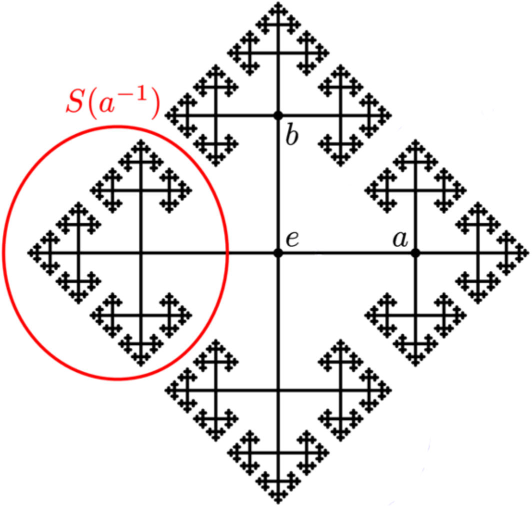
The Group F_2



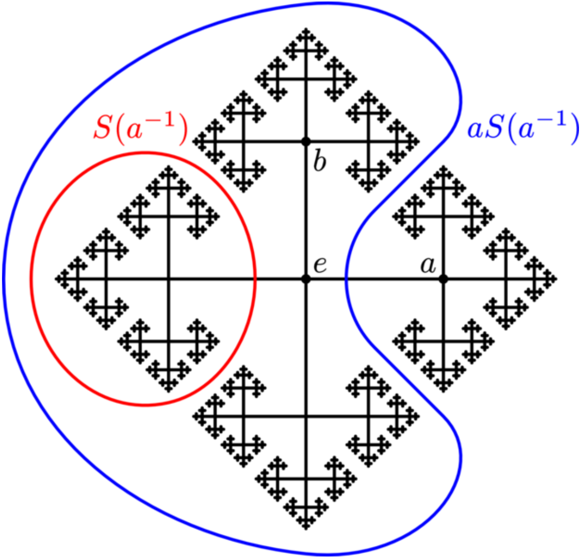
The Partition



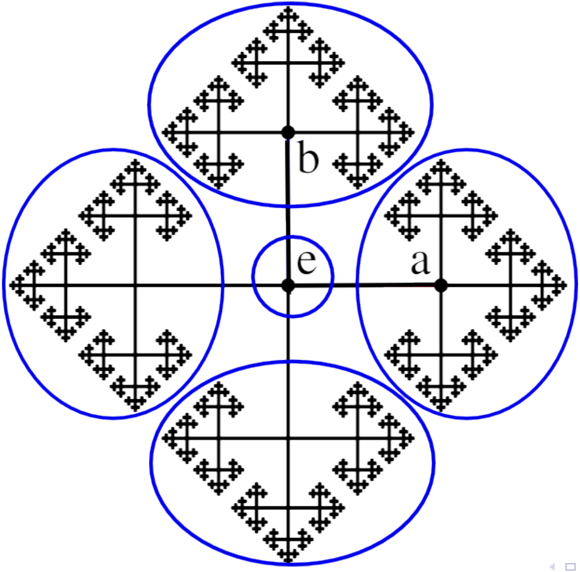
One Piece



After multiplication by *a*



Two Copies!!!



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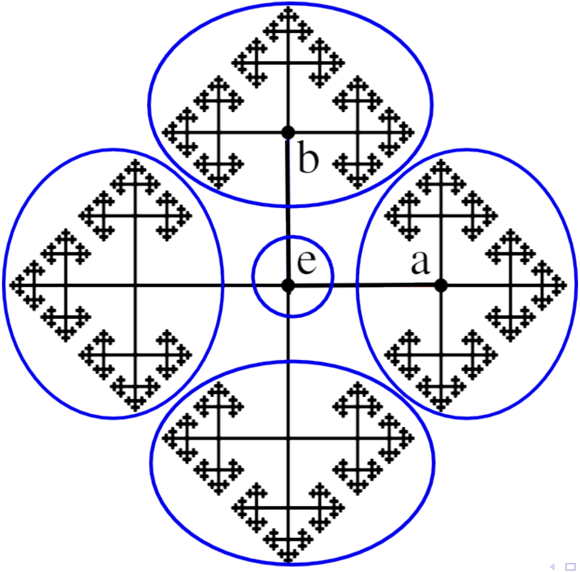
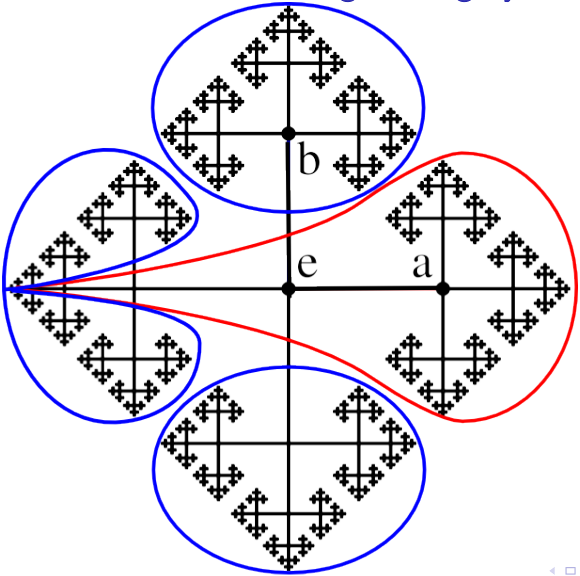


... and a dot

Slight Surgery



Incredible

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Can we find a structure like F_2 inside the solid sphere?

The Sphere and the Shell

$$\begin{aligned} \mathcal{B}_3 &= \left\{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 \leq 1\right\} \\ \mathcal{S}_2 &= \left\{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = 1\right\} \end{aligned}$$

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We shall just restrict ourselves to finding a *paradoxical decomposition* in the spherical shell. Very easy to lift it to the entire sphere.

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Does there exist a copy of F_2 sitting inside this group? Yes

F_2 Inside the Group of Rotations

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Claim

Let σ be the rotation about the x-axis by 1 radian and τ be the rotation about the y axis by 1 radian. Then, σ and τ generate a free group.

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... try it out. ☐

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Great. Lets call this group $\mathcal{F}$.

Recall that $\mathcal{F}$ can be decomposed into $\mathcal{A}_1$,$\mathcal{A}_2$,$\mathcal{A}_3$,$\mathcal{A}_4$ such that

$$\mathcal{F} = \mathcal{A}_1 \cup \sigma\mathcal{A}_2 = \mathcal{A}_3 \cup \tau\mathcal{A}_4$$

The *Almost*-Sphere

Each non-trivial $\rho \in \mathcal{F}$ has precisely two fixed points on $\mathcal{S}_2$. Let D be the set of all fixed points of $\mathcal{F}$.

$$\tilde{\mathcal{S}}_2 = \mathcal{S}_2 - D$$

Roadmap

- Transform $\mathcal{S}_2$ to $\tilde{\mathcal{S}}_2$
- Double $\tilde{\mathcal{S}}_2$, forming two copies of it
- Transform each copy of $\tilde{\mathcal{S}}_2$ back to $\mathcal{S}_2$

Step 1 and 3 are inverses of each other; we shall cover that after we look at step 2.

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Orbits of $\tilde{\mathcal{S}}_2$

Claim

For any $\rho \in \mathcal{F}$, $p \in D$ *if and only if* $\rho(p) \in D$.

Proof.

If $\sigma(p) = p$, then $\rho\sigma\rho^{-1}(\rho(p)) = \rho(p)$. ☐

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Hence, the group $\mathcal{F}$ acts on $\tilde{\mathcal{S}}_2$ as well. Look at the set of orbits and pick a set R of representatives (using the *Axiom of Choice*).

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Consider the following sets:

$$A_i = \{\rho(R) \ : \ \rho \in \mathcal{A}_i\} \qquad , \qquad i = 1, 2, 3, 4$$

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Proof.

If $\rho_1(r) = \rho_2(r)$ for $\rho_1 \neq \rho_2$, then r will be a fixed point of $\rho_1^{-1}\rho_2$. □

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\implies A_1 \cup \sigma(A_2) &= \tilde{\mathcal{S}}_2
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□

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- The number of bad θ's is countable. Pick a good angle $\bar{\theta}$ and let $\rho = \rho_{\bar{\theta}}$.

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Proof.

Wlog, assume $m > n$. If $\rho^m(d_1) = \rho^n(d_2)$, then $\rho^{m-n}(d_1) = d_2$, which isn't possible by our choice of ρ. ☐

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$$\begin{aligned} \rho(A) &= \bigcup_{n=1}^{\infty} \rho^n(D) = A - D \\ \implies \rho(A) \cup B &= \mathcal{S}_2 - D = \tilde{\mathcal{S}}_2 \end{aligned}$$

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- The Banach-Tarski Paradox

Appetiser to the Proof
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Doubling the Sphere
- Preliminaries
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That completes the proof of the Banach-Tarski Paradox.

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  (an easier example of it is the Vitali Set. Wiki it)
- Any proof of the Banach-Tarski paradox *has* to use the Axiom of Choice.

PJ!

What is a nice anagram of Banach-Tarski?

PJ!

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Banach-Tarski Banach-Tarski

Thank You