Topology

Mid-Semester Examination, Semester IV

23rd February, 2006

50 marks

1. Define nowhere dense sets, sets of first and second category. State Baire category theorem. (5 marks)

2. Give an example to show that continuous functions need not be open maps (i.e map open sets to open sets) (7 marks)

3. Show that an arbitrary product of Hausdorff spaces is Hausdorff, and that an arbitrary product of T_1 spaces is T_1. (8 marks)

4. Let f_i be a countable collection of continious functions, $f_i : \mathbb{R} \to \mathbb{R}$. Let $X_i \subset \mathbb{R} \times \mathbb{R}$ be their graphs with the subspace topology. Show that $\mathbb{R}^2 \neq \bigcup X_i$. (10 marks)

5. Define a component of a space to be a maximal connected subspace (i.e not contained in any other connected subspace). Define locally connected space X to be a space which has a basis of connected neighbourhoods. Show that a compact locally connected space has a finite number of components. (8 marks)

6. Let X and Y be topological spaces. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps such that $f \circ g = id_Y$, the identity map on Y. Prove the following:

   - f is surjective and g is injective.
   - Y has the quotient topology determined by f.
   - g maps Y homeomorphically onto a subspace of X.
   - If X is Hausdorff, so is Y

   (7 marks)

7. Let $\mathbf{P^{n-1}}$ be the set of one dimensional subspaces of $\mathbb{R}^n$. Show that this is bijective to the quotient space $(\mathbb{R} - (0))/\mathbb{R}^*$, where the group $\mathbb{R}^*$ of non-zero real numbers act on $\mathbb{R}^n - (0)$ be scalar multiplication. Give $\mathbb{R}$ the standard topology and the spaces involved the natural topology coming from this. Describe open sets in $\mathbf{P^{n-1}}$ with the quotient topology. Show that $\mathbf{P^{n-1}}$ is compact. (15 marks)