Discrete Mathematics

End-Semester Examination, Semester I

April, 2005

1. How many integers x in the range $1 \leq x \leq 1000$ are not divisible by $2, 3$ or 5? [2 marks]

2. Suppose there are m girls and n boys in a class. What is the number of ways of arranging them in a line so that all the girls go together? [2 marks]

3. Show that the number of integer partitions of $2n$ into three parts such that the sum of any two parts is greater than the third, is equal to the number of integer partitions of n with exactly 3 parts. [3 marks]

4. When $k \geq 0$, let $f(n, k)$ be the number of $k-$element subsets of $[n]$ which do not contain two successive integers. Show that

$$f(n, k) = f(n-2, k-1) + f(n-1, k)$$

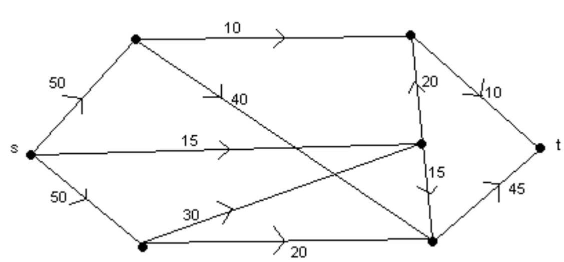
Let $F_k(x)$ be the generating function of the numbers $f(n, k)$ for fixed k. Find a recursion for $F_k(x)$ and deduce that

$$f(n, k) = \binom{n-k+2}{k}$$

[4 marks]

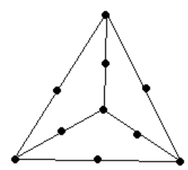
5. Suppose that a tree T contains a vertex of degree k. Show that it has atleast k leaves. [3 marks]

6. Compute a maximum $s - t$ flow in the following network and prove that it is indeed maximum.



[4 marks]

7. Prove or disprove: The graph below has a perfect matching:



[3 marks]

8. Let T be a tree on atleast 3 vertices. Define the graph T^2 as teh graph with vertex set $V(T)$ and two distinct vertices are connected by an edge in T^2 iff their distance in T is atmost 2. Prove or disprove: T^2 is 2-connected. [4 marks]

9. Fix a positive even integer $n = 2m$. A chain is a sequence of sets in inclusion: $S_1 \subset S_2 \subset \ldots \subset S_l$ where each $S_i \subseteq [n]$. Call such a chain summetric if it contains (for some number k) one set of size k, one set of size $k+1$, $\ldots$, one set of size $n-k$ and no other sets.

   - For $o \leq k < m$, show that the k-size subsets of $[n]$ can be matched into $k+1$ size subsets such that if S is a k-size subset matched to the $k+1$-size subset T, then $S \subset T$ [3 marks].
   - Use this association to construct a partition of the set of all subsets of $[n]$ as a union of several disjoint symmetric chains [2 marks].
   - A collection of subsets of $[n]$ is said to be an anti-chain if it contains no two different sets S, T such that $S \subset T$. Prove that the maximum size of an anti-chain is equal to the minimum number of symmetric chains needed to partition the set of all subsets of $[n]$. [3 marks]

10. Let G be the group of all automorphisms of the complete bipartite graph $K_{3,3}$ and let v be any vertex of the graph. Compute the size of the stabilizer of v under G and hence the size of G. [3 marks]

11. How many coloured disks can be constructed by dividing one side of the disc into five equal sectors and colouring two sectors red, two sectors white and one sector blue? [4 marks]