Complex Analysis

Mid-Semester Examination, Semester IV

22nd February, 2006

100 marks

1. Find all points in $\mathbb{C}$ where $f(z) = |z|^2$ is differentiable. (5 marks)
2. Let n be an integer. Set $\omega = e^{2\pi i/n}$. Calculate $(2+\omega)(2+\omega^2)\cdots(2+\omega^{n-1})$. (5 marks)
3. Find a power series expansion of $1/(z^2+1)$ around $z = 1$ and calculate its radius of convergence. (6 marks)
4. Does the formula $\sin(z+w) = \sin z \cos w + \cos z \sin w$ hold for all complex numbers z, w. (8 marks)
5. The power series $f(z) = \sum z^n = 1 + z + z^2 + \cdots$ clearly has radius of convergence 1.
   - Is there any point on the unit circle where the power series converges?
   - Find all points w on the unit circle such that $\lim_{z\to w} f(z)$ exists as a finite number as z approaches w from within the open unit disc?

   (6 marks)
6. Set $L^- := \{z \in \mathbb{C} | x = 0, Im(z) \geq 0\}$. Let $U = \mathbb{C}\backslash L^-$.
   - Define a branch of $\log z$ on U.
   - Is U a maximal open set for this branch?
   - Find two complex numbers z_1, z_2 such that $\log z_1 + \log z_1 \neq \log z_1 z_2$ for this branch
   - Compare thi branch with the principal branch (defined over the complement of the negative real axis)
   - Find a harmonic conjucate of $\log\sqrt{x^2+y^2}$ on U.

   (20 marks)
7. - Find the cross ratio $(0, i, -1, -i)$.

- Find a Mobius transform which sends the open unit disk to the upper half plane.

(6 marks)

8. Find all points in $\mathbb{C}_\infty$ where $\tan z := \sin z / \cos z$ can be defined as a differentiable function mapping $\mathbb{C}_\infty$. (12 marks)

9. For any function $f : \mathbb{C} \to \mathbb{C}$, define

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \;\&\; \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial z} + i\frac{\partial f}{\partial y}\right)$$

Verify that f is analytic iff $\frac{\partial f}{\partial z} = 0$ and moreover that the derivative f' equals $\frac{\partial f}{\partial \overline{z}}$. (10 marks)

10. Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ be two convergent power series in some disk D of positive radius. If there is a sequence $\{z_k\}$ of distinct points in D converging to 0 such that $f(z_k) = g(z_k)$ for all k, then show that $f = g$. (8 marks)

11. Does there exist an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that f sends a pair of perpendicular lines to the unit circle. (7 marks)

12. Let p be a polynomial function on $\mathbb{C}$. Let U be an open connected set in $\mathbb{C}$ and f an analytic function on U such that $|f| = |p|$ on U. Show that f is also a polynomial function. *(Hint: Infact, f is constant times p)* (12 marks)

13. Consider the power series $f(z) = \sum a_n z^n$ where $a_n = 1$ if n is a power of 2, $a_n = 0$ otherwise. Thus

$$f(z) = 1 + z + z^2 + z^4 + z^8 + \ldots$$

- Verify that the radius of convergence of this power series is 1
- Show that for any point w on the unit circle, $\lim_{z \to w} f(z)$ does not exist as a finite number. *(Hint: First look radially in the direction $\theta = (\frac{k}{2^m})2\pi$)*

(12 marks)