

Calculus III

End Semester

December 2005

1. Calculate the arc-length of the curve parametrized by $\mathbf{f}(t) = (2t, \ln t, t^2)$. (6 marks)
2. Find two potential functions of $\mathbf{F}(x, y, z) = (2xy, x^2 + z \cos y, \sin y)$. (6 marks)
3. Calculate the line integral of the vector field $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$ along the square in \mathbb{R}^2 described by $\{(x, y) | \max\{|x|, |y|\} = 1\}$ and oriented anticlockwise. (14 marks)
4. Calculate the line integral of the vector field $\mathbf{G}(x, y) = (y\mathbf{i} - x\mathbf{j})/\sqrt{x^2 + y^2}$ along the unit circle oriented clockwise. (6 marks)
5. Find an open subset U of \mathbb{R}^4 or prove that none exists such that the vector field $\mathbf{F}(x, y, z, w) = (x, y, z, yw)$ has a potential function on U . (6 marks)
6. Find the equation of the plane tangent to the surface $x^3 + y^3 + z^3 = 2$ in \mathbb{R}^3 at the point $(0, 1, 1)$. (4 marks)
7. Find a continuous piecewise smooth simple closed C in \mathbb{R}^2 (use u, v as coordinates of \mathbb{R}^2) and a differentiable parametrisation form C together with its interior D to the cylinder $S = \{(x, y, z) | 0 \leq z \leq 1, x^2 + y^2 = 1\}$ via some function such that the normal at any point of S is pointing inwards. Moreover,
 - identify $\mathbf{F}(C)$ in simple terms.
 - for the anticlockwise orientation on C , describe the induced by the orientation on the projection of $F(C)$ in the x - y plane.(14 marks)
8. Find the surface area of the graph $z = x\sqrt{3} + y^2$ lying over the region $\{(x, y) | 0 \leq y \leq 1, 0 \leq x \leq y\}$ in \mathbb{R}^2 . (4 marks)
9. Let C be the circle of radius $1/\sqrt{\pi}$ centered at $(11, -2)$ and oriented anticlockwise. Prove that the line integral of the vector field

$$F(x, y) = (2y + 3xy - \sin(e^{x^2-1}), 2x^2 + 2x + \cos(e^y - y^3))$$

along C is the same as the average value of x on the disc enclosed by C . Using symmetry considerations or otherwise obtain a numerical answer. (8 marks)

10. Show that the (flux) integral over of the vector field

$$F(x, y, z) = (ye^z, x^2 + \cos xz, x - y)$$

through any cube in \mathbb{R}^3 is zero. (6 marks)

11. The cylinder $x^2 + y^2 = 1$ in \mathbb{R}^3 intersects the plane $x + y + z = 9$ in an ellipse. Show that for any parametrisation $\alpha : I \rightarrow \mathbb{R}^3$ of the ellipse and any differentiable vector field from \mathbb{R}^3 to \mathbb{R}^3 such that

$$\nabla \times F = (1 + y + z, 1 + z + x, 1 + x + y)$$

Then show that the line integral $\oint F \cdot d\alpha$ on the ellipse is $\pm 21\pi$. (9 marks)

12. Find the unique differentiable function $y(x)$ on \mathbb{R} satisfying

- $y'' + 2y' - 15y = 0$
- $y'(0) = 1$ & $y''(0) = 2$

(6 marks)

13. Give an infinite set of differentiable functions $y(x)$, satisfying

- $xy' = 2y$
- $y(1) = 1$

(6 marks)