Calculus II

End-Semester Examination, Semester II

April, 2005

1 Lesson

State exactly the following theorems:

1. the *Taylor* formula with the *Lagrange* reminder,
2. the theorem on differentiation of limits and limits of derivatives,
3. and both of *Fubini*'s theorems.

2 Exercises

1. Give an example of a continuous function f such that $\int_0^{+\infty} f(u)du$ exists but we don't have $\lim_{n\to+\infty} f(u) = 0$.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\lim_{t\to+\infty} f(t) = l \in \mathbb{R}$. Show that:
$$\lim_{a\to+\infty} \int_0^{+\infty} \frac{af(t)}{a^2+t^2} dt = \frac{\pi l}{2}$$
3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as:
$$f(x) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$
   (a) Is f continuous at $(0,0)$?
   (b) Show that $f \in C^1(\mathbb{R}^2, \mathbb{R})$.
   (c) Compute the second order partial derivatives of f. What can you conclude?
4. Let $f(x,y) = x^2 + xy + y^2 - 3x - 6y$.
   (a) Show that f admits only one extremum on $\mathbb{R}^2$.
   (b) Write $f(x,y)$ as a sum of 2 squares and a constant. Conclude that f attains -9 as global minimum.

5. Let D be the domain of $\mathbb{R}^2$ delimited by the lines $x = 0, y = x + 2$ and $y = -2$.
   (a) Draw D and compute directly $I = \int\int_D (x - y) dx\, dy$.
   (b) Consider the change of variable $u = x + y$ and $v = x - y$. What is teh domain corresponding to D? Use it to compute I

6. Let $p > 0$ and
$$D = \left\{(x, y) \in \mathbb{R}^2 : y^2 - 2px \leq 0 \text{ and } x^2 - 2py \leq 0\right\}$$
   (a) Draw D.
   (b) Justify the use of the change of variable $x = u^2 v$ and $y = uv^2$, compute the jacobian determinant and find the domain corresponding to D.
   (c) Use it to compute $\int\int \exp\left(\frac{x^3+y^3}{xy}\right) dx\, dy$

7. Let $0 < a \leq b, 0 < c \leq d$ and
$$D = \left\{(x, y) \in \mathbb{R}^2 : ax^2 \leq y \leq bx^2, \frac{c}{x} \leq y \leq \frac{d}{x}\right\}$$
   Justify the usage of teh change of variables $u = \frac{y}{x^2}$ and $v = xy$ and compute the area of D.

8. Let $V = \left\{(x, y, z) \in \mathbb{R}^3 : -1 \leq z \leq 1, x^2 + y^2 \leq z^2 + 1\right\}$. Draw V and compute its volume.

9. Compute the volume of intersection of two cylinders with axis (Ox) and (Oy) and with same radius $R > 0$.

3 Γ function and volume of spheres

Let define the following Γ function
$$\Gamma(s) = \int_0^{+\infty} x^s \exp(-x) \frac{dx}{x}$$

1. For which values of the parameter s the quantity $\Gamma(s)$ is defined, and for which valuse of this parameter the corresponding integral is *absolutely* convergent?

2. Use integration by part to obtain the the following formula (functional equation of the Γ function.
$$s \cdot \Gamma(s) = \Gamma(s + 1)$$
   wherever $\Gamma(s)$ is defined.

3. Compute $\Gamma(1)$ and deduce $\Gamma(n)$ for all $n \in \mathbb{N}$.

4. Express the integral $\int_{-\infty}^{+\infty} \exp(-x^2)dx$ as a value of Γ using change of variable.

   From now on we fix some $n \in \mathbb{N}$

5. Express
$$\int_{\mathbb{R}^n} \exp(-x_1^2 - \ldots - x_n^2)dx_1 \ldots dx_n$$
in terms of Γ using one of *Fubini's* theorems.

   We denote by $\mathbf{S}$ the sphere of radius 1 centered atthe origin of the vector space $\mathbb{R}^n$ and $\mathbf{B}$ the corresponding ball of radius 1. $R.\mathbf{S}$ and $R.\mathbf{B}$ denote the sphere and ball respectively of radius R.

   We define *integration over the sphere* $R.\mathbf{S}$ of a function f
$$\int_{R.\mathbf{S}} f d\sigma$$
as the derivative, whenever defined, at $r = R$ of teh following one parameter integral
$$\int_{r.\mathbf{B}} f(x_1, x_2, \ldots, x_n)dx_1 dx_2 \ldots dx_n$$
We define $Vol(R.\mathbf{S}) = \int_{R.\mathbf{S}} 1 d\sigma$ and $Vol(R.\mathbf{B}) = \int_{R.\mathbf{B}} 1 dx_1 dx_2 \ldots dx_n$.

6. Prove tht $Vol(R.\mathbf{B}) = R^n Vol(\mathbf{B})$, and that $Vol(\mathbf{B}) = Vol(\mathbf{S})/n$ and that $Vol(R.\mathbf{S}) = R^{n-1} Vol(\mathbf{S})$, when $R > 0$.

7. Prove that
$$\int_{\mathbb{R}^n} \exp(-x_1^2 - \ldots - x_n^2)dx_1 dx_2 \ldots dx_n = \int_{r=0}^{+\infty} Vol(r.\mathbf{S}) \exp(-r^2)dr$$

8. Express the above right hand side in terms of a value of Γ and $Vol(\mathbf{S})$.

9. Express $Vol(\mathbf{S})$ and $Vol(\mathbf{B})$ in terms of values of Γ using 5 and 8. Simplify in terms of $\Gamma(1/2)$ in the two distinct cases of whether n is even or odd.

10. Using the definition of $Vol(\mathbf{B}) = \pi$ for $n = 2$, compute $\Gamma(1/2)$.

    Extra questions(bonus points):

11. Prove infinite differentiability of Γ on its domain. Compute the integral expressions for iterated derivatives of Γ.

12. For f in class C^2, express $\log(f)''$.

13. Show that the denominator of this expression is strictly positive.

14. Remind why $\log(u)^2 + \log(v)^2 \geq 2\log(u)\log(v)$ for u, v strictly positive, and what are cases of equality?

15. Deduce for s in the domain of Γ

$$\left(\int_0^{+\infty} \log(u)^2 u^s \exp(-u)\frac{du}{u}\right) \cdot \left(\int_0^{+\infty} v^s \exp(-v)\frac{dv}{v}\right) >$$

$$\int_0^{+\infty}\int_0^{+\infty} \log(u)\log(v) \cdot (uv)^s \cdot \exp(-u-v)\frac{du}{u}\frac{dv}{v}$$

16. Deduce that $\log(\Gamma)'' > 0$, that $\log(\Gamma)$ is convex and that Γ is thus also convex.