Calculus III

Mid-Semester Examination, Semester III

20 September, 2005

100 Marks

1. Find the area enclosed by the triangle formed by the points $(1, 2, 3)$, $(2, 3, 4)$ and $(-1, 7, 0)$ in $\mathbb{R}^3$. [4 points]

2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ be points in $\mathbb{R}^n$.

   (a) Find the point closest to $\mathbf{u}$ in the plane P parametrized as $\mathbf{v}+t\mathbf{w}+s\mathbf{x}$.

   (b) What is the distance between them?

   (c) Give the parametric form of the line through u that intersects P and is perpendicular to the plane.

   [12 points]

3. Find three vectors in $\mathbb{R}^4$ having the angle $\pi/3$ between them pairwise. [5 points]

4. The intesection of the cylinder $x^2+y^2 = 1$ in $\mathbb{R}^3$ with the surface $x \sin z = y \cos z$ is a helix (curve spiralling upwards). in the following cases, find differentiable parametrizations of the helis such that the tangent vector at $(1, 0, 0)$ is the given vector:

   - $(0, 1, 1)$
   - $(0, 0, 0)$

   [8 points]

5. Find 2 sequences of points v_k and u_k approaching the origin of the cuspidal curve defined parametrically by (t^2, t^3) in $\mathbb{R}^2$ such that $\lim_{k\to\infty} \frac{v_k - u_k}{\|v_k - u_k\|} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. [6 points]

6. (a) Show that for any curve in $\mathbb{R}^2$ parametrized as $t \mapsto (r_1(t), r_2(t))$, the curvature at the point on the curve corresponding to t is the absolute value of

$$\frac{r_1' r_2'' - r_1'' r_2'}{((r_1')^2 + (r_2')^2)^{3/2}}$$

(b) Parametrize the ellipse $(x/a)^2 + (y/b)^2 = 1$ and find the maximum and minimum values of the curvature on it. Identify the points.

[16 points]

7. $\mathbb{R}^1 \xrightarrow{u} \mathbb{R}^2 \xrightarrow{\mathbf{f}} \mathbb{R}^n$ be a differentiable function where $u' > 0$. Show that $\mathbf{f}$ and $\mathbf{f}u$ have the same unit speed reparametrizations. [6 points]

8. The function $f(x, y) = \frac{x^3 y^3}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and is assigned the value 0 at the origin. Does that make it continuous? [6 points]

9. Calculate the patial derivatives of $x^{1/5}y^{1/5}$ at the origin using the first principles. Is the function differentiable at the origin? [8 points]

10. Calculate the directions derivative of $f(x, y, z) = e^x + \log(y^2 z^2 + 1)$ along the unit vector parallel to $(1, 0, 1)$. [4 points]

11. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x, y) = (e^{x+y}, \sin x + \cos y)$. Finda parametrized curve $f : \mathbb{R}^1 \to \mathbb{R}^2$ such that $g := F \circ f$ passes through $(1, 1)$ and has tangent vector $(0, 1)$ there. [8 points]

12. Calculate the curl of the function $F(x, y, z) = y\mathbf{i} - x\mathbf{j}$. Is it the gradient of any scalar function $\mathbb{R}^3 \to \mathbb{R}$? [6 points]

13. Let $\mathbf{r} : \mathbb{R}^3 \to \mathbb{R}^3$ be the position function $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Set $r := \|\mathbf{r}\|$. Calculate the gradient of $1/r$ (away from the origin) in terms of $\mathbf{r}$ and r. [6 points]

14. Given differentiable function $F = (f_1, f_2, f_3)$ and $G = (g_1, g_2, g_3)$ from $\mathbb{R}^3$ to $\mathbb{R}^3$, let $H = (h_1, h_2, h_3)$ be teh composite $F \circ G$. With x, y, z as coordinate functions on $\mathbb{R}^3$, express $\partial h_2/\partial z$ in terms of the partials of F and G.[5 points]