Analysis I

Mid-Semester Examination, Semester III

23 September, 2005

1. Show that the set
$$\{nx - m | n, m \in \mathbb{Z}\}$$
is dense in $\mathbb{R}$ if and only if x is irrational.

2. Suppose
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ for all } |z-a| < R$$
Show that for any complex number b such that $|b-a| < R$,$f(z)$ has a power series expansion about the point b which is valid for $|z-b| < R - |b-a|$. Conclude that the function $f(z)$ is analytic on $\{z : |z-a| < R\}$.

3. Find a sequence $\{f_n\}$ of Riemann-integrable real-valued functions on [0,1] which converges pointwise to 0 but for which
$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq 0$$

4. Let $f : [a, b] \to \mathbb{R}$ be a continuous function taking positive values. Let M be the maximum value of f on $[a, b]$. Show that:
$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^n dx \right)^{\frac{1}{n}} = M$$

5. Suppose that the series $\sum a_n$ of positive terms is divergent. Construct a divergent series $\sum b_n$ of positive terms such that $\lim_{n\to\infty}(b_n/a_n) = 0$.