Higgs bundles on Riemann surfaces

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- Congratulations to N&S for their beautiful theorem!
- Great impact in Mathematics & Physics
- Certainly great influence on my work starting with my Doctorate thesis.

- N&S theorem deals with study of real reps of fundamental group of cpt smooth surface S in $U(n)$

$$\rho : \pi_1(S) \longrightarrow U(n)$$

<u>More generally</u> we can consider

$$\rho : \pi_1(S) \longrightarrow G \text{ — Compact Lie group}$$

<u>Character variety of moduli of representations</u>

$$\mathcal{R}(S, G) := \mathrm{Hom}(\pi_1(S), G)$$

- <u>Related to $G^{\mathbb{C}}$-bdles</u> : Let $X = (S, J)$ — cpx str.

$M(X, G^{\mathbb{C}})$ — moduli space of semistable $G^{\mathbb{C}}$-bdles.

<u>thm (N-S, Ramanathan)</u>

$$\mathcal{R}(S, G) \underset{\text{real analytic}}{\overset{\text{homeo}}{\cong}} M(X, G^{\mathbb{C}})_0$$

— Vanishing Free part of top. class in $\pi_1(G)$

• Now, it's important to study repr. in <u>non-compact groups</u>.

G — non-cpt <u>reductive</u> Lie group.

Difference with compact case

$$\mathrm{Hom}(\pi_1(S), G) \big/ G$$

is generally <u>non-Hausdorff</u> (only Hausdorff when G abelian)

Have to consider <u>reductive reps</u>:

$$\rho : \pi_1(S) \to G$$

<u>reductive</u> $\iff$ $\overline{\rho(\pi_1(X))}^{Z}$ reductive

$$\iff \begin{tikzcd} \pi_1(X) \arrow[r, "\rho"] \arrow[rr, bend right] & G \arrow[r, "Ad"] & GL(\mathfrak{g}) \end{tikzcd}$$

completely reducible

$$\mathcal{R}(S, G) = \mathrm{Hom}^{red}(\pi_1(X), G) \big/ G$$

$$\left(\text{if } G \text{ complex} = \underbrace{\mathrm{Hom}(\pi_1(X), G) /\!/ G}_{GIT}\right)$$

$\mathcal{R}(S,G)$ — real (analytic / algebraic) variety
— complex if G is complex.

- The simplest non-abelian example for a real non-compact group is $G = SL(2,\mathbb{R})$

Character variety $\mathcal{R}(S, SL(2,\mathbb{R}))$ deeply linked to the geometry of the surface

$$\rho : \pi_1(X) \longrightarrow SL(2,\mathbb{R})$$

$d = d(\rho)$ = Euler class of ρ

($\rho \leftrightarrow E_\rho$ flat $SL(2,\mathbb{R})$-bdle
Topological class of $E_\rho \in \pi_1(SL(2,\mathbb{R})) \cong \mathbb{Z}$)

Milnor ('58)

$$|d| \leq g-1 \qquad g = \text{genus of } X$$

$$\mathcal{R}_d = \{\rho \in \mathcal{R}(S, SL(2,\mathbb{R})) : d(\rho) = d\}$$

$\mathcal{R}_{g-1}$ = Teichmüller space

Fuchsian reps. I'll come back to this later.

- Similarly to the compact case, reps of $\pi_1(S)$ in a non-compact Lie group are related to holomorphic objects: Higgs bdles!

G-Higgs bdles

G semisimple real Lie group (more generally reductive)

$H \subset G$ maximal cpt subgroup.

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition

$\perp$ Killing form of $\mathfrak{g}$.

$$Ad_G|_H : H \longrightarrow GL(\mathfrak{h}) \oplus GL(\mathfrak{m})$$

adjoint isotropy

$$\iota : H^{\mathbb{C}} \longrightarrow GL(\mathfrak{m}^{\mathbb{C}})$$ isotropy

G – Higgs bdle: (E, φ)

E — $H^{\mathbb{C}}$-bdle on $X = (S, J)$ — Riemann surface

$\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$

associated $\mathfrak{m}^{\mathbb{C}}$-bdle

canonical line bdle of X.

Remarks

- G compact $\quad H = G \quad , \mathfrak{m} = 0$

  G – Higgs bdle $\longleftrightarrow$ $G^{\mathbb{C}}$–bdle.
  $(\varphi = 0)$

- G complex

  $G = H^{\mathbb{C}} \quad , \quad \mathfrak{g} = \mathfrak{h} + i\mathfrak{h} \Rightarrow \mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{g}$

  G - Higgs bdle $(E, \varphi) \longleftrightarrow$ E G-bdle

  $\varphi \in H^0(E(\mathfrak{g}) \otimes K)$

  adjoint bdle

These are the objects introduced by Hitchin in

... original papers.

He first considered: $G = SL(n, \mathbb{C})$ $\left(GL(n,\mathbb{C})\right)$

$E \longleftrightarrow$ rk n vector bdle V
with $\Lambda^n V \cong \mathcal{O}$.

$\varphi : V \longrightarrow V \otimes K$
with $\mathrm{Tr}(\varphi) = 0$.

- We're interested in non-cpt real groups specially real forms of a cpx semisimple Lie group.

- there are stability criteria for G-Higgs bdls

$\mathcal{M}(X, G)$ — Moduli space of semistable (or polystable) G-Higgs bdls.

Analogue of N-S theorem is:

<u>Thm</u> (Non-Abelian Hodge theory)

$$\mathcal{R}(S, G) \underset{\text{homeo}}{\cong} \mathcal{M}(X, G)$$

- In contrast with N–S Theorem, where one has to prove an existence thm for non-linear gauge-theoretic PDE's (Donaldson's proof of the N–S theorem based on Atiyah–Bott's approach) here one has to prove two existence thms for non-linear PDE's !!

→ First theorem solves Hitchin equations for a metric on the bundle:

Let (E,φ) b a G-Higgs bdle. We look for a reduction of structure group $h \in \Gamma(E(H^{\mathbb{C}}/H))$

$$\boxed{F_h + [\varphi, \varphi^{*_h}] = 0}$$

Curvature of Chern connectn on E defined by h

$$*_h : \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \longrightarrow \Omega^{0,1}(E(\mathfrak{m}^{\mathbb{C}}))$$

conjugation defined by h & conj. $\Omega^{1,0} \to \Omega^{0,1}$

Theorem

(E,φ) admits a sol. to Hitchin eqns $\iff$ (E,φ) polystable

- $G = SL(2,\mathbb{C})$: Hitchin '87 (Dim'l reduction of $SU(2)$ instantons on $\mathbb{R}^4$)
- G complex : Simpson '88
- G real : Bradlow, G, Gothen, Mundet.

[Remark : If G compact $\Rightarrow \varphi = 0$ & Hitchin eqn is just $F_h = 0$ & hence we obtain the N-S Theorem.]

(Story is over here!)

In the non-cpt case $\varphi \neq 0$

To get from a sol. to Hitchin eqns to a rep $\pi_1(S) \to G$

Consider the G-connection

$$D = \underset{\underset{H}{\uparrow}}{\nabla_h} + \underbrace{\varphi + \varphi^{*h}}_{\underset{M}{\uparrow}}$$

∇_h : Chern conn.

Hitchin eqns $\Longrightarrow$ D flat $\Longrightarrow$ $\rho : \pi_1(S) \to G$ holonomy rep. which is reductive

- The converse is provided by an existence theorem for <u>Harmonic</u> reduction of structure group on flat bundle

E_G — smooth G-bdle

D — Flat G connection on E_G.

Given:

$$h \in \Gamma\left(E_G(G/H)\right)$$

$$D = \underset{\substack{\uparrow \\ \mathfrak{h}}}{\nabla_h} + \underset{\substack{\uparrow \\ \mathfrak{m}}}{\Psi}$$

h <u>harmonic</u> $\iff \nabla_h^* \Psi = 0$

$D \Longleftrightarrow \rho : \pi_1(S) \to G$ holonomy rep of D.

$h \longleftrightarrow \tilde{h} : \tilde{S} \to G/H$

$\pi_1(S)$ — equiv. ($\pi_1(S)$ acting via ρ on G/H)

$\nabla_h^* \Psi = 0 \iff \tilde{h}$ harmonic in the usual sense.

minimising energy $E(\tilde{h}) = \int_S |d\tilde{h}|^2$

Theorem (Donaldson Corlette)

Let (E_G, D) be a flat G-bundle & let $\rho: \pi_1(S) \to G$ be the holonomy rep.
(E_G, D) admits a harmonic metric
$\Longleftrightarrow$ ρ is reductive

Donaldson '87: $G = SL(2,\mathbb{C})$
(Corlette: '88: General

Generalizes Eells–Sampson (1964) same year as N–S!

→ This correspondence is the starting point for complex analytic & algebraic methods to study the varieties $R(S,G)$ by means of studying instead $\mathcal{M}(X,G)$, in particular the topology of $R(S,G)$.

→ Main applications: Use of Morse Theory or Localization at $\mathbb{C}^*$-fixed points $\lambda(E,\varphi) = (E, \lambda\varphi)$ to compute the topology of these moduli spaces: Hitchin started this for $G = SL(2,\mathbb{C})$

→ Very interesting phenomena even for the counting of components of $\mathcal{M}(G)$ for certain type of groups.

To illustrate this we go back to simplest example.

$$\begin{array}{ccccc} G & = & SU(1,1) & \cong & SL(2,\mathbb{R}) \\ \cup & & \cup & & \cup \\ H & = & S(U(1)\times U(1)) & \cong & SO(2) \\ & & \wr\wr & & \\ & & U(1) & & \end{array}$$

$$H^{\mathbb{C}} = \mathbb{C}^* \cong SO(2,\mathbb{C})$$

$$\mathfrak{sl}(2,\mathbb{C}) = \underset{\substack{\wr\wr \\ \mathbb{C}}}{\mathfrak{so}(2,\mathbb{C})} + \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$$

G-Higgs bdle on X

$(L, \beta, \gamma) \longleftrightarrow (L \oplus L^*, Q)$ — $SO(2,\mathbb{C})$-bdl

$$\varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

$L \to X$ line bdle

$\beta : L^{-1} \to L \otimes K \iff \beta \in H^0(L^2 \otimes K)$

$\gamma : L \to L^{-1} \otimes K \iff \gamma \in H^0(L^{-2} \otimes K)$

Topological invariant

$d = \deg L$

Milnor inequality

$|d| \leq g-1$ (assume $g \geq 1$)

Pf:) Suppose $d > g-1 \Rightarrow \gamma = 0 \Rightarrow \varphi = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \Rightarrow$ L φ-inv, $\deg L > 0$ violates semistability

Assume $d \geq 0$

Let $\mathcal{M}_d$ be the moduli space of $SL(2,\mathbb{R})$ Higgs bdls with top. inv $= d$.

In his first paper Hitchin gave a explicit description of $\mathcal{M}_d$

Here I will just look at $|d| = g-1$

Assume $d = g-1$ $\quad (\mathcal{M}_d \cong \mathcal{M}_{-d})$.

$$\underset{\text{semistable}}{d = \deg L = g-1} \Rightarrow \gamma \not\equiv 0 \Rightarrow L^2 \overset{\gamma}{\cong} K$$

$L \cong K^{1/2}$ (Theta characteristic) 2^{2g} choices

$\beta \in H^0(X, K^2)$.

$\longrightarrow$

$\mathcal{M}_{g-1} = 2^{2g}$ copies of $H^0(X, K^2) \cong \mathbb{C}^{3g-3}$

$\mathcal{M}_{g-1} \cong R_{g-1}$ — Teichmüller space.

(More about this type of groups in the talks of Biquard & Collier)

We have a map

$$\mathcal{M}(SL(2,\mathbb{R})) \longrightarrow H^0(X, K^2)$$
$$\left(L \oplus L^*, \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right) \longmapsto \det\varphi = -\beta\gamma$$

"Teichmüller" components are sections of this map: $\beta \longmapsto \left(K^{1/2} \oplus K^{-1/2}, \varphi = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}\right)$

More generally there is the:

Hitchin map

$\mathbf{H}^{\mathbb{C}} \longrightarrow GL(\mathfrak{m}^{\mathbb{C}})$ isotropy rep

$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$ — algebra of $H^{\mathbb{C}}$-inv. polynomials on $\mathfrak{m}^{\mathbb{C}}$

$p_1, \ldots, p_r$ basis of $\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$

Let $n_i = \deg p_i \quad i = 1, \ldots, r$

There is a map

$$h : \mathcal{M}(G) \longrightarrow B(G) := \bigoplus_{i=1}^{r} H^0(X, K^{n_i})$$
$$(E, \varphi) \longmapsto (p_1(\varphi), \ldots, p_r(\varphi)).$$

More intrinsically,

$\mathfrak{a} \subset \mathfrak{m}^c$ maximal abelian subalgebra

$W(\mathfrak{a})$ (baby) Weyl group

$\mathbb{C}[\mathfrak{m}^c]^{H^c} = \mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})}$ Chevalley

$$B(G) = H^0(X, \mathfrak{a} \otimes K / W(\mathfrak{a})).$$

Hitchin considered this when G is complex

- Completely alg. integrable system: Fibres at a "generic" point in $B(G)$ is a generalised Prym variety
  - Hitchin '87 for classical groups using spectral covers of X.
  - More general Donagi, Scognamillo, Faltings, ... (cameral covers of X.
- Hitchin constructed a section of h: whose image is a component of $\mathcal{M}(G_{split})$: Hitchin–Teichmüller components.

$\hookrightarrow$ story.

- What can we say when G is real?

Regarding a section to Hitchin map:

theorem (G-Peón-Ramanan).

Let $h: \mathcal{M}(G) \to B(G)$ be the Hitchin map. There is a section $s: B(G) \to \mathcal{M}(G)$ factoring through $\mathcal{M}(\hat{G}) \to \mathcal{M}(G)$, where $\hat{G}$ is the maximal split subgroup of G. The image is a component iff G is $\mathbb{R}$-split.

As for the study of the fibres:

- L. Schaposnik uses spectral covers for some classical groups.

- A. Peón uses cameral covers for general groups. Fact: fibre is "abelian" iff G is quasi-split (G complex, G split real form, $SU(n,n)$, $SU(n,n+1)$, ...).

For some "quaternionic" real groups Hitchin-Schaposnik show that fibre is a moduli of rank 2 vector bdles.

→ This phenomena had been observed first by E. Franco in his thesis on Higgs bdles over elliptic curves

Hyperkähler structure when G is complex

$\mathcal{M}(G)$ — cpx structure J_1 coming from X

$\mathcal{R}(G)$ — cpx structure J_2 coming from G.

Define $J_3 = J_1 J_2$.

g — hyperkähler metric on $\mathcal{M}(G)$ given by sol. to Hitchin equations
(hyperkähler quotient).

Symplectic forms

$$\omega_i(\cdot,\cdot) = g(J_i\cdot,\cdot) \quad i=1,2,3.$$

Let $G_{real} \subset G$ real form i.e.

$$G_{real} = G^\sigma \quad , \quad \sigma : G \to G \text{ antih. inv.}$$

There is a morphism

$$\mathcal{M}(G^\sigma) \longrightarrow \mathcal{M}(G)$$

image is $\underline{J_1\text{-hol.}}$ & $\underline{\omega_2 \ \& \ \omega_3 \text{ Lagrangian}}$

examples of (B, A, A)-branes.

$\Leftrightarrow$ $\underline{J_1\text{-hol Lagrangian}}$ w.r.t. to $(J_1, \Omega_1 = \omega_2 + i\omega_3)$

One way to see this is to show that the image is in the fixed point loci of the involution

$$(E, \varphi) \longmapsto (a(E), -a(\varphi))$$

where $a \in Out_2(G)$ corresponding to σ. (shown in joint work with Ramanan)

In particular, the moduli of G-bdles. appears in the fixed locus for the invol.

$$(E, \varphi) \longmapsto (E, -\varphi)$$

J_2- & J_3- hol. Lagrang.

- To give examples of other types of branes (A, B, A), (A, A, B), one considers a conjugation on $\alpha : X \to X$ (Klein surface) & conj. $\sigma : G \to G$ and considers the involutions

$$\mathcal{M}(G) \longrightarrow \mathcal{M}(G)$$
$$(E, \varphi) \longmapsto (\alpha^{*}\sigma(E), \pm\alpha^{*}\sigma(\varphi))$$

studied in joint work with Biswas & Hurtubise

– also studied by Baraglia & Schaposnik.

(See also Schaffhauser & Wilkin et al)

$(\alpha, \sigma, \pm)$-real G-Higgs bdls

(E, φ) G-Higgs bdle

$$\begin{tikzcd} E \arrow[r, "\sigma_E"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\alpha"] & X \end{tikzcd}$$

$$\sigma_E(eg) = \sigma_E(e)\sigma(g)$$

$$\sigma_E^2 = 1 \qquad (c \in Z(G) \text{ pseudo-real})$$

$$\tilde{\sigma} : E(\mathfrak{m}^{c}) \otimes K \to E(\mathfrak{m}^{c}) \otimes K$$

$$\tilde{\sigma}(\varphi) = \pm\varphi.$$

Orbifold fdl group

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(X) \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & G \arrow[r] & G_\pm \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1 \end{tikzcd}$$

$$G_\pm = G \times_{\pm} \mathbb{Z}/2$$

σ

cpt

$\theta = \sigma\tau$