Non-Perturbative Hyperkähler Manifolds

P. Boalch (CNRS & Orsay)

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cf. also surveys arXiv: 1203 (Hyperkähler)
1309 (Poisson)

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Boalch, Markman 1995

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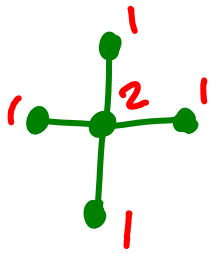
Eg 

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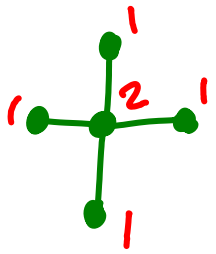
Eg 

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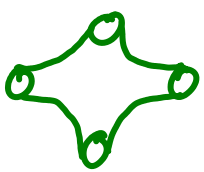
$\mathcal{M}$( , GL_2)

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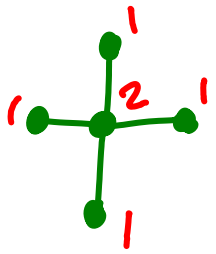
Eg 

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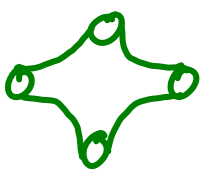
$\mathcal{M}$( , GL_2)

$\parallel$

$\mathcal{M}_{Betti} \cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \} \subset \mathbb{C}^3$

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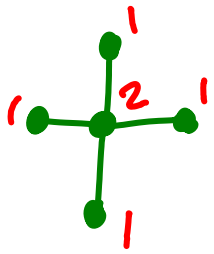
Eg 

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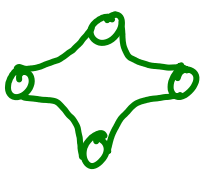
$\mathcal{M}\left(\text{  }, GL_2\right)$

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what about $\Rightarrow$ Eguchi-Hanson $\subset$?

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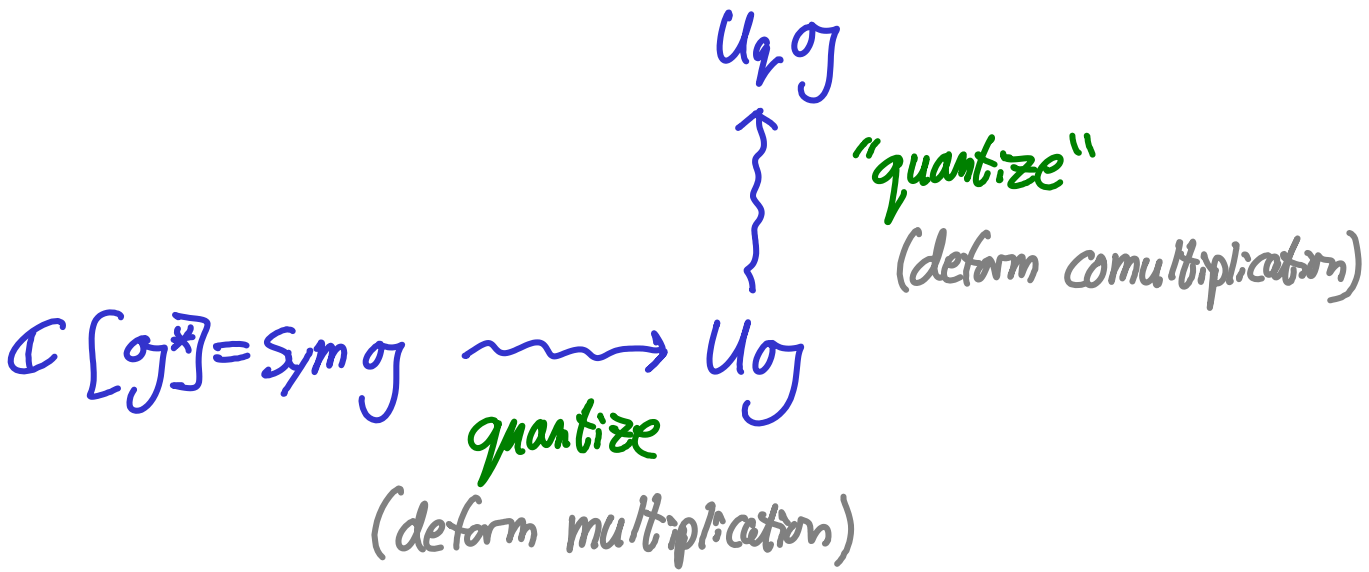
$\left(\frac{A}{z^2} + \frac{B}{z}\right) dz$ $\Rightarrow$ Poisson Lie group underlying $U_q \mathfrak{g}$

$\mathbb{C}[[g^*]] = \mathrm{Sym}\, g \rightsquigarrow U g$

quantize

(deform multiplication)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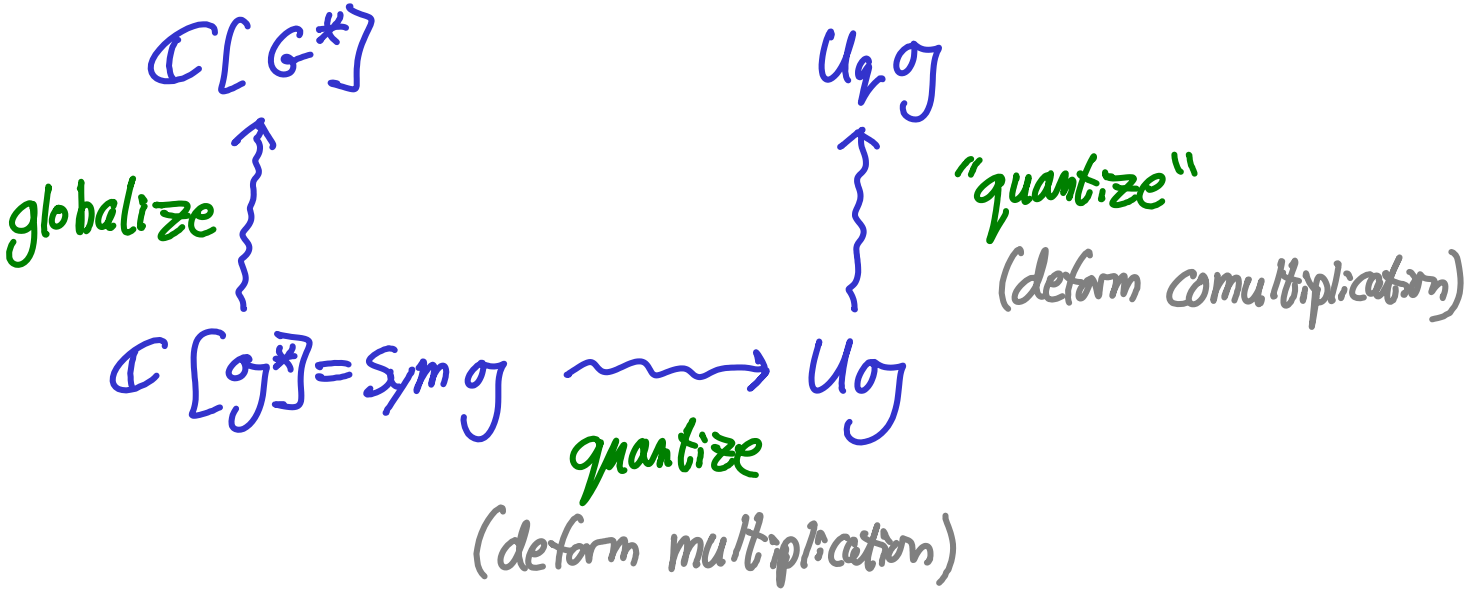
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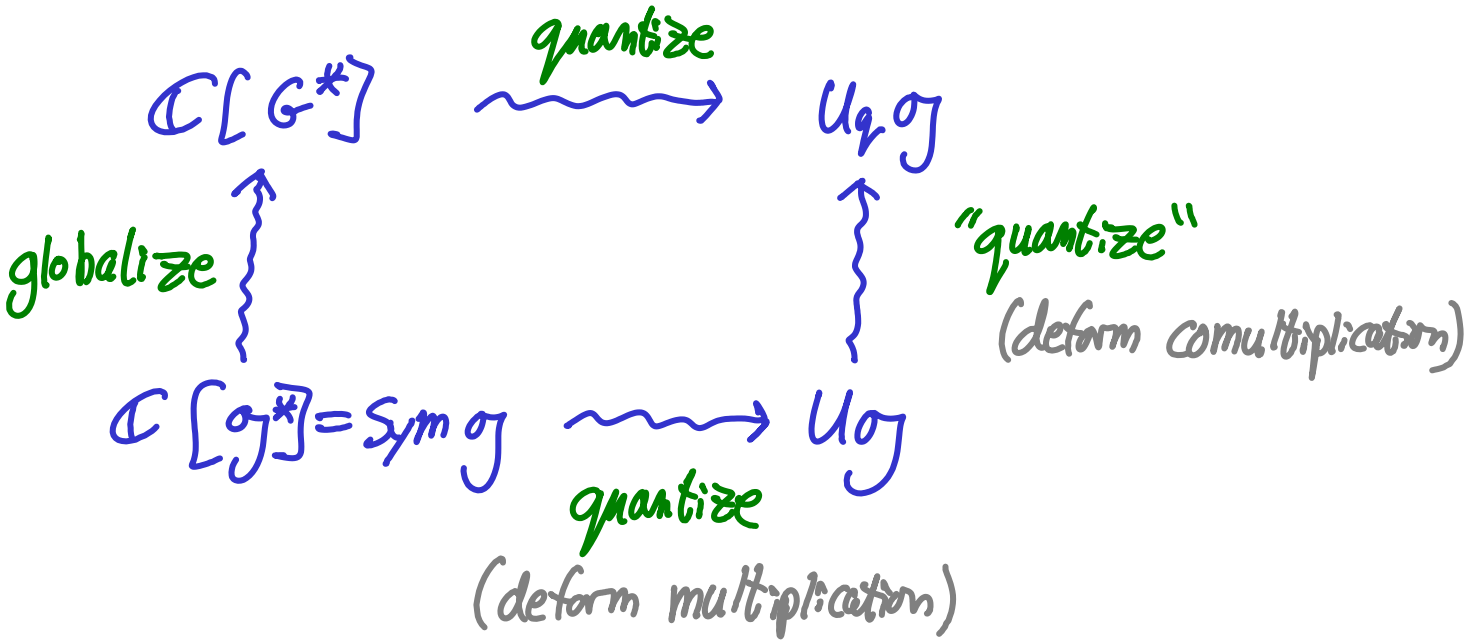
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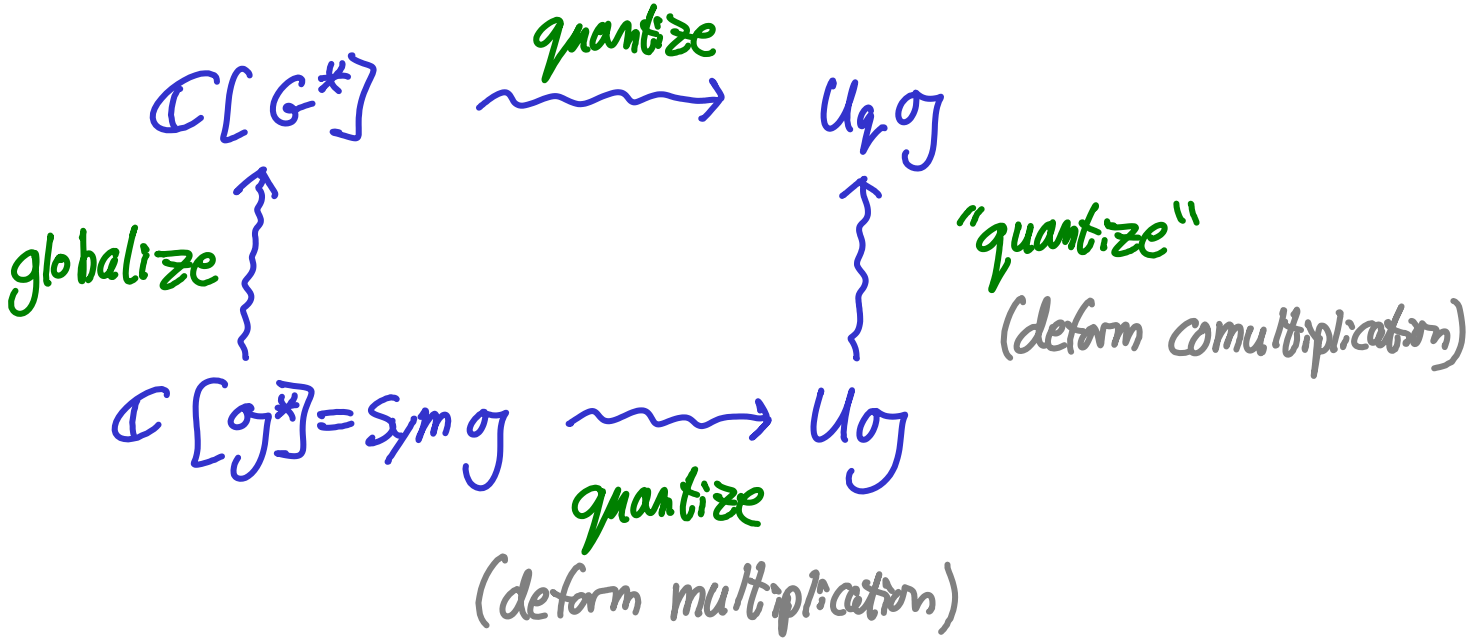
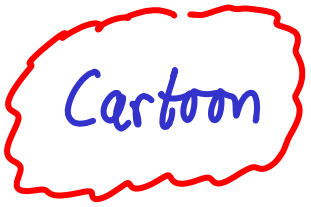


Thm (2001) G^* is the space of monodromy/Stokes data of connections $\left(\frac{A}{z^2} + \frac{B}{z}\right)dz\Big|_{\text{unit disc}}$ $A \in \mathfrak{t}_{\text{reg}}$ fixed, $B \in \mathfrak{g} \cong \mathfrak{g}^*$

and the desired nonlinear Poisson structure appears this way


Cartoon

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Hamiltonian geometry

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Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

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∞-d Ham^n geometry

e.g. connections on C^∞ bundles / Riemann surfaces

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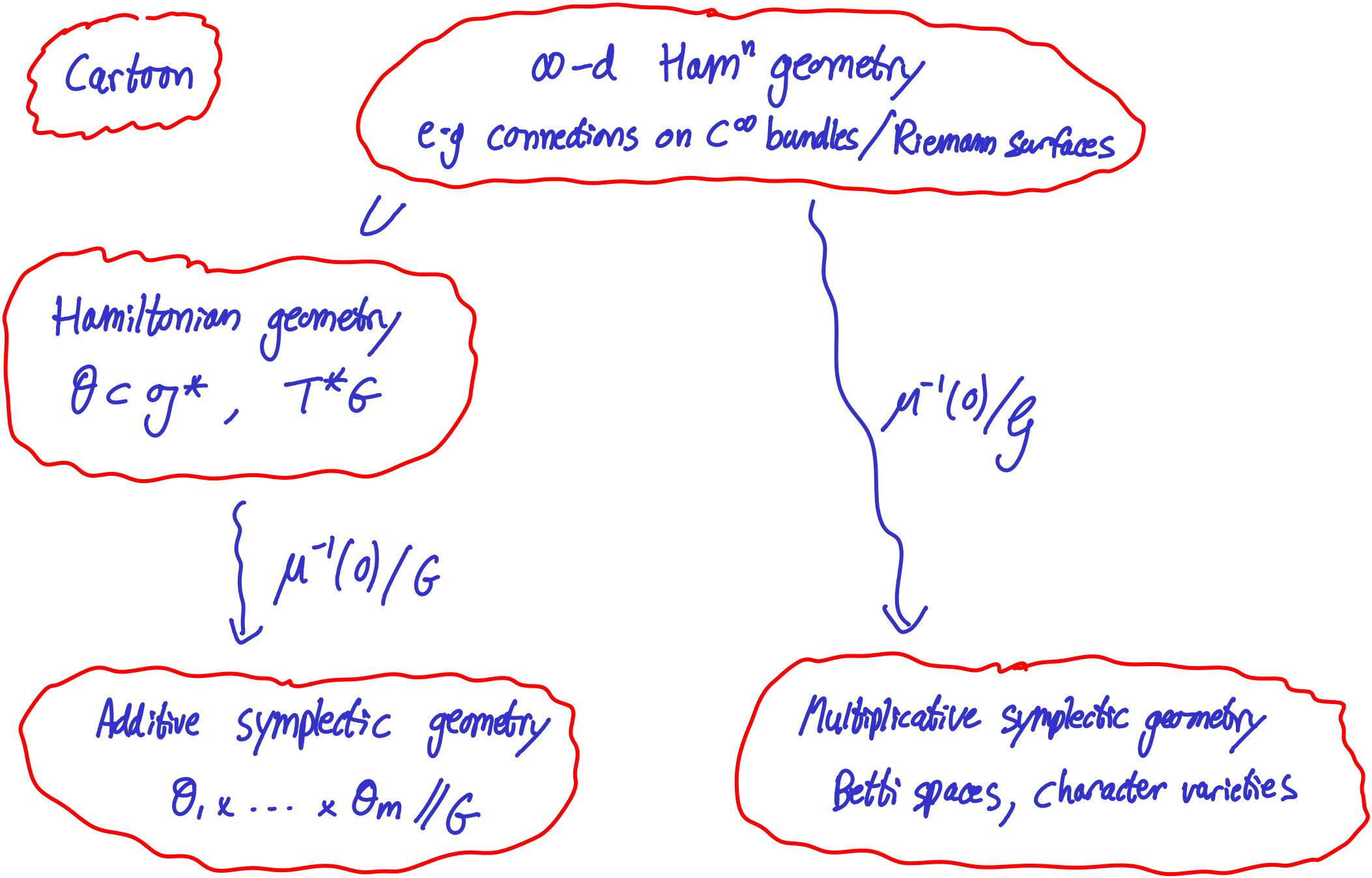
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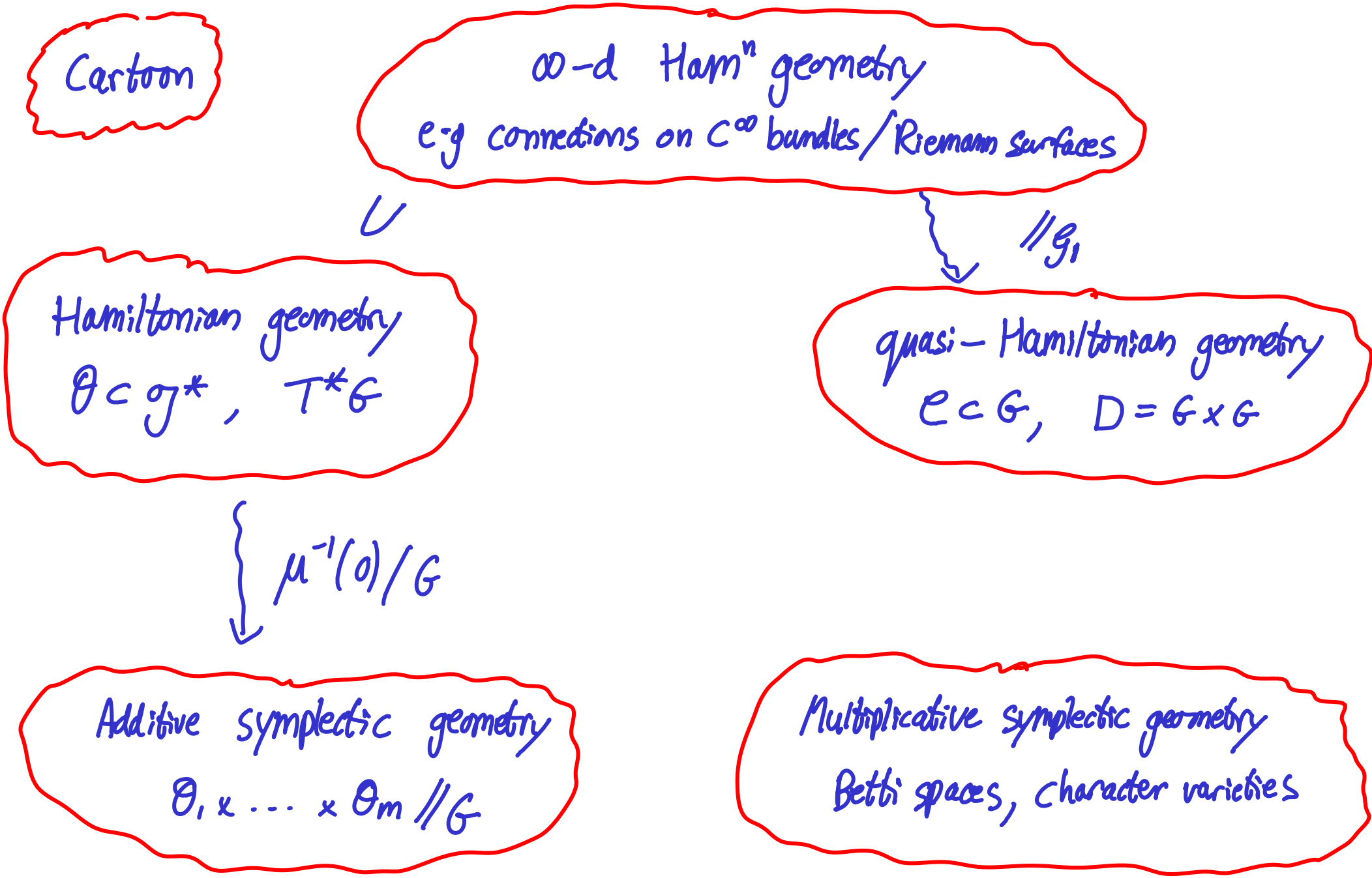
Betti spaces, character varieties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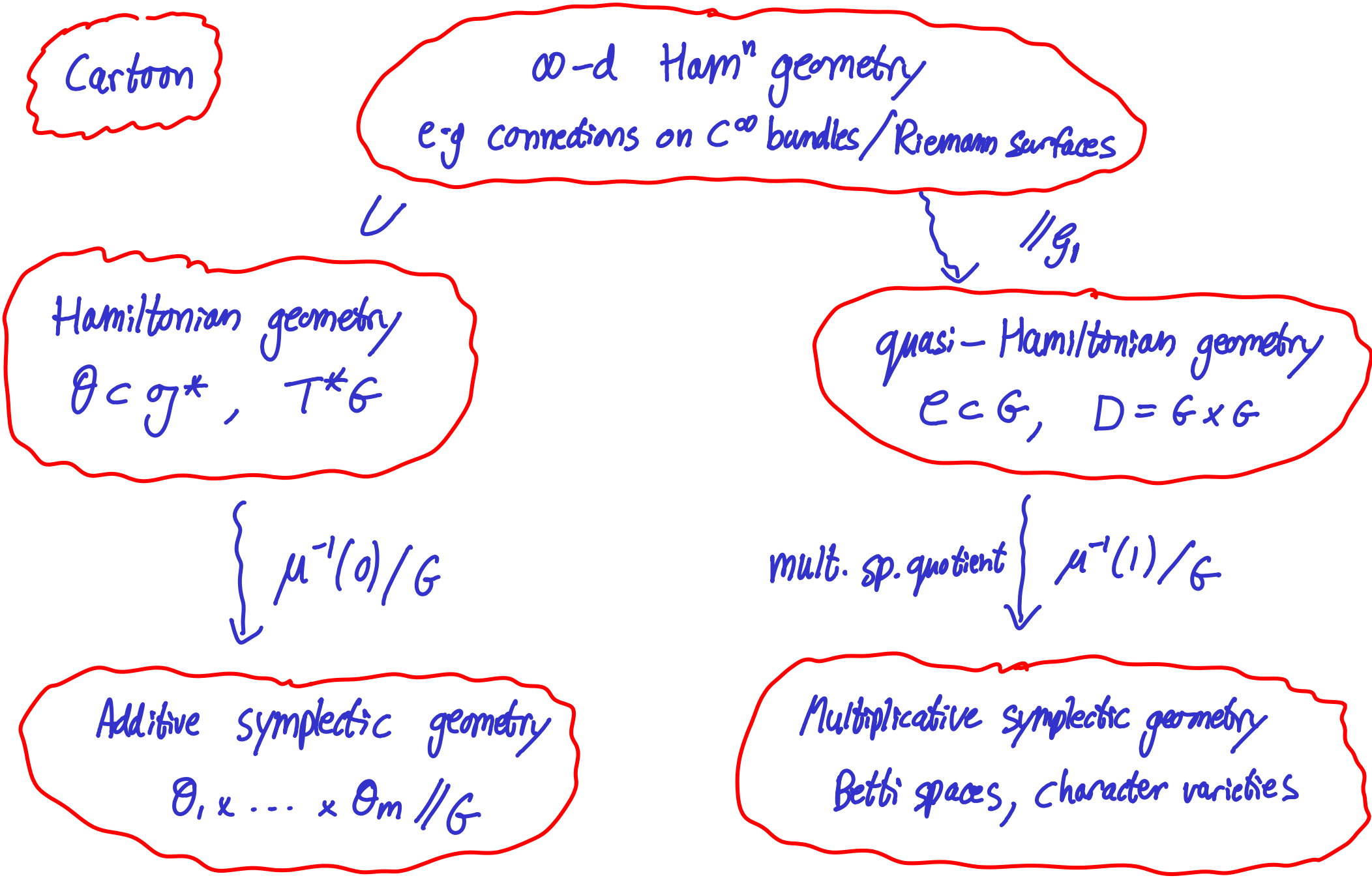
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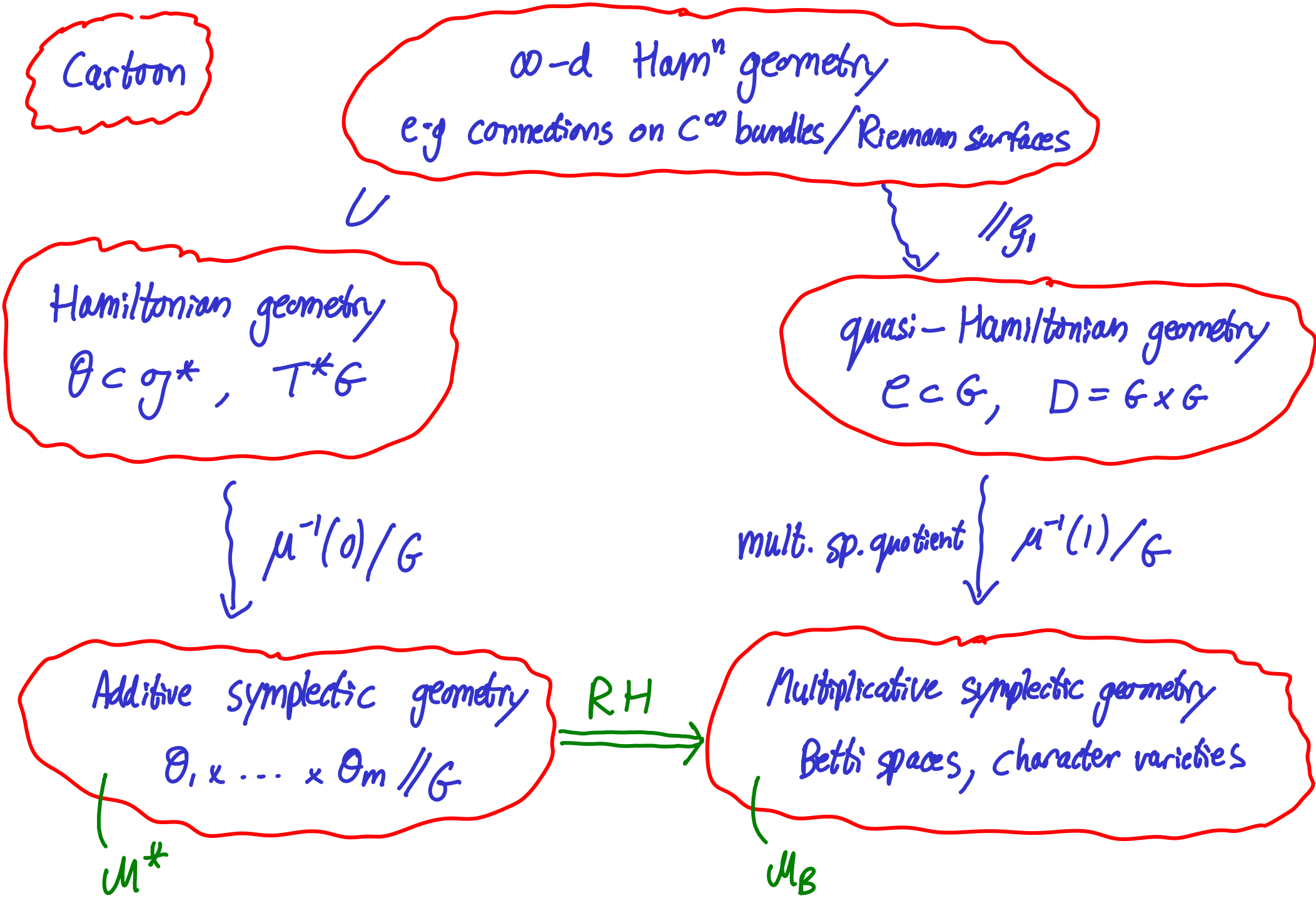
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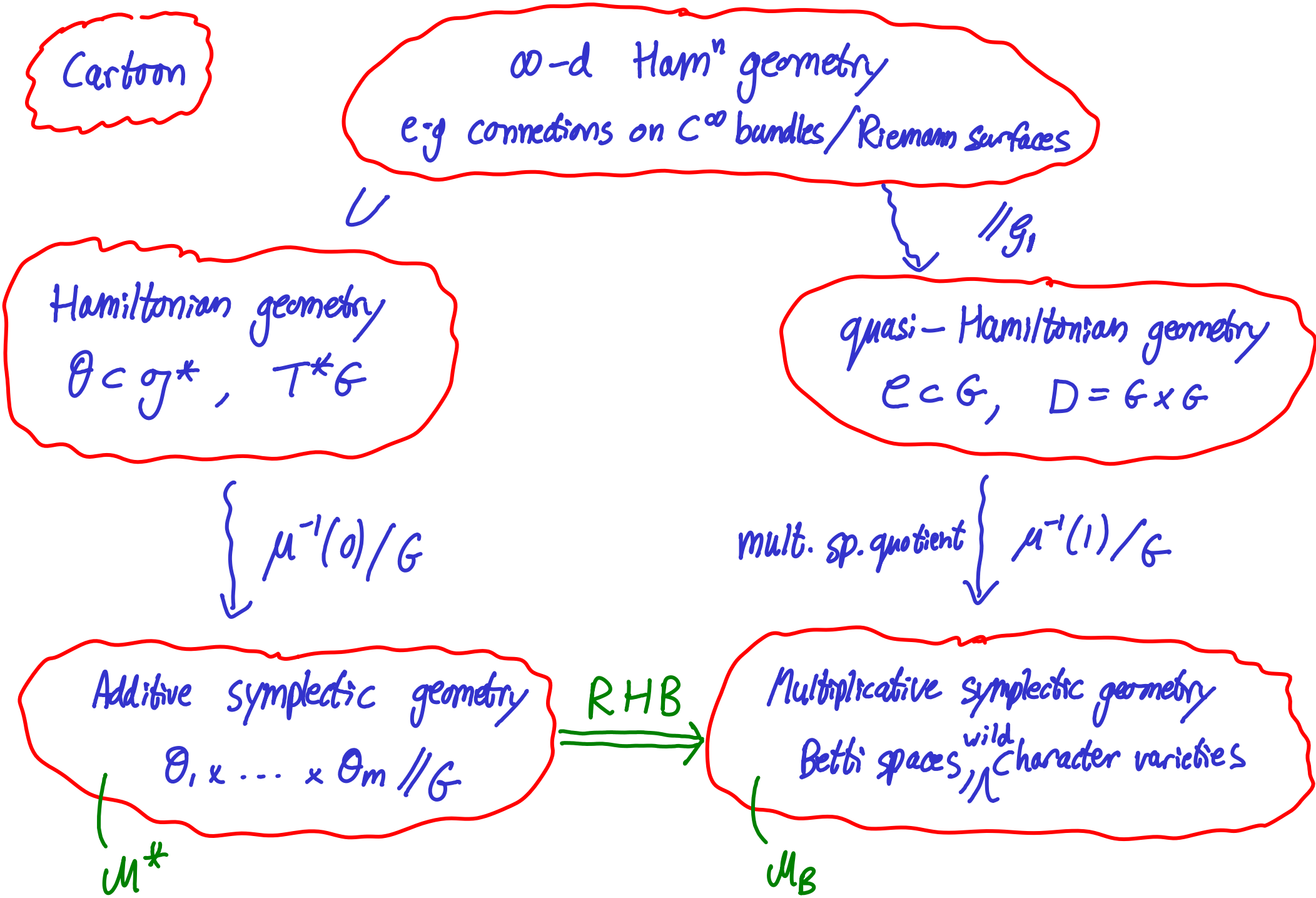
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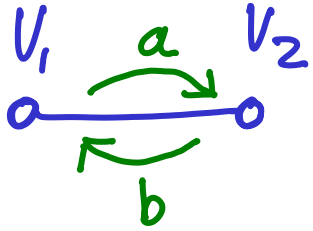
$\Gamma =$ 

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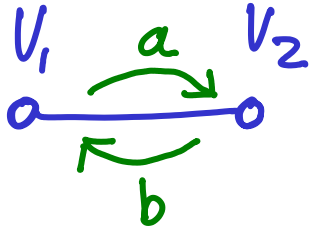
$\Gamma =$ 

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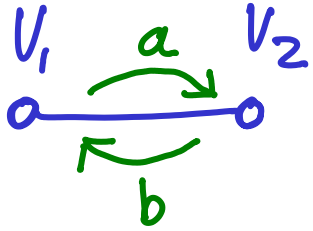
$\Gamma =$ 

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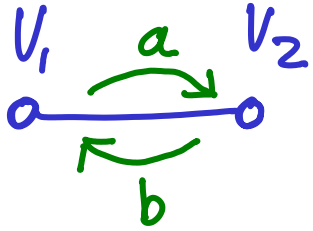
$\Gamma =$ 

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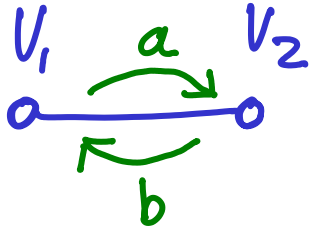
$\Gamma =$ 

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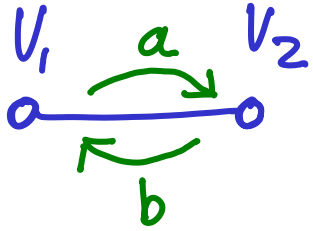
$\Gamma =$ 

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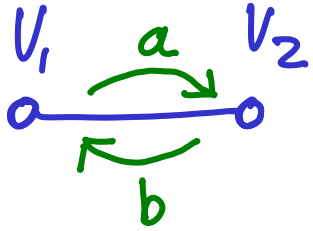
$\Gamma =$ 

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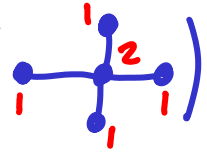
E.g. Mult. Quiver Var. $\left(\right.$ 

$\left.\right) \cong \{xyz + x^2 + y^2 + z^2 = ax + by + cz + d\}$

Qn Suppose $\Gamma =$ or etc

then what is $\mathrm{Rep}^*(\Gamma, V)$?

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$\cup$

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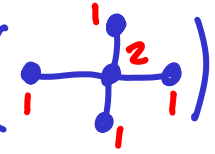
E.g. Mult. Quiver Var.  $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

SPECIMEN ALGORITHMI SINGVLARIS.

Auctore
L. EVLERO.

1.

Conſideratio fractionum continuarum, quarum vſum vberrimum per totam Analyſin iam aliquoties oſtendi, deduxit me ad quantitates certo quodam modo ex indicibus formatas, quarum natura ita eſt comparata, vt ſingularem algorithmum requirat. Cum igitur ſumma Analyſeos inuenta maximam partem algorithmo ad certas quasdam quantitates accommodato

6. Haec ergo teneatur definitio ſignorum ( ), inter quae indices ordine a ſiniſtra ad dextram ſcribere conſtitui; atque indices hoc modo clauſulis incluſi inpoſterum denotabunt numerum ex iſtis indicibus formatum. Ita a ſimpliciſſimis caſibus inchoando, habebimus:

$$(a) \quad = a$$
$$(a,b) \quad = ab + 1$$
$$(a,b,c) \quad = abc + c + a$$
$$(a,b,c,d) \quad = abcd + cd + ad + ab + 1$$
$$(a,b,c,d,e) = abcde + cde + ade + abe + abc + e + c + a$$

etc.

"Euler's continuant polynomials"

G. G. Stokes 1857

VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments.* *By* G. G. Stokes, M.A., D.C.L., Sec. R.S., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *May* 11, 1857.]

In a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series," printed in the ninth volume of the *Transactions* of this Society, I succeeded in developing the integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$ in a form which admits of extremely easy numerical calculation when m is large, whether positive or negative, or even moderately large. The method there followed is of very general application to a class of functions which frequently occur in physical problems. Some other examples of its use are given in the same paper; and I was enabled by the application of it to solve the problem of the motion of the fluid surrounding a pendulum of the form of a long cylinder, when the internal friction of the fluid is taken into account *.

These functions admit of expansion, according to ascending powers of the variables, in series which are always convergent, and which may be regarded as defining the functions for all values of the variable real or imaginary, though the actual numerical calculation would involve a labour increasing indefinitely with the magnitude of the variable. They satisfy certain linear differential equations, which indeed frequently are what present themselves in the first instance, the series, multiplied by arbitrary constants, being merely their integrals. In my former paper, to which the present may be regarded as a supplement, I have employed these equations to obtain integrals in the form of descending series multiplied by exponentials. These integrals, when once the arbitrary constants are determined, are exceedingly convenient

Wild Character Varieties

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Σ compact Riemann Surface $\Longrightarrow$ $\mathcal{M}_B = Hom(\pi_1(\Sigma), G)/G$

symplectic variety

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Σ compact Riemann Surface $\Longrightarrow$ $\mathcal{M}_B = Hom(\pi_1(\Sigma), G)/G$ — symplectic variety

$\| \int_{RH}$

$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on $\Sigma \}/_{isom}$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \ldots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$ — symplectic variety

$\| \int$ RH

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Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Σ compact Riemann surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$\Longrightarrow$ Poisson variety

$\mathcal{M}_B^{tame} = \mathrm{Hom}(\pi_1(\Sigma^\circ), G)/G$

$\| \int$ RH

$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on Σ° with reg. sings $\}/_{isom}$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Poisson scheme (∞-type)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \ldots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$\| \int$ RHB

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Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\Big\| \int$ RHB

$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on Σ° with irreg. types $\underset{\sim}{Q}$ $\}/\text{isom}$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

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$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on $\Sigma^\circ \}$ / isom with irreg. types $\underset{\sim}{Q}$

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Cartan subalg.

$Q_i \in \tau_i \subset \mathfrak{g}((z_i))$

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$\| \int$ RHB

$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on Σ° with irreg. types $\underset{\sim}{Q}$ $\}/$ isom

$\nabla \cong d + Q_i + \Lambda_i \frac{dz_i}{z_i} + \text{holom.}$

e.g. $Q_i \in \mathfrak{t}((z_i)) \subset \mathfrak{g}((z_i))$ ($\mathfrak{t}$ Cartan subalg.) $\subset \mathfrak{g}$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Wild Riemann surface $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Rightarrow$ wild character variety

Σ compact Riemann surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$

$\Rightarrow$ $\mathcal{M}_B$

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Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. $(Disc, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$Q = A/\mathbb{Z}^k$, $\quad A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $\quad a \neq b$

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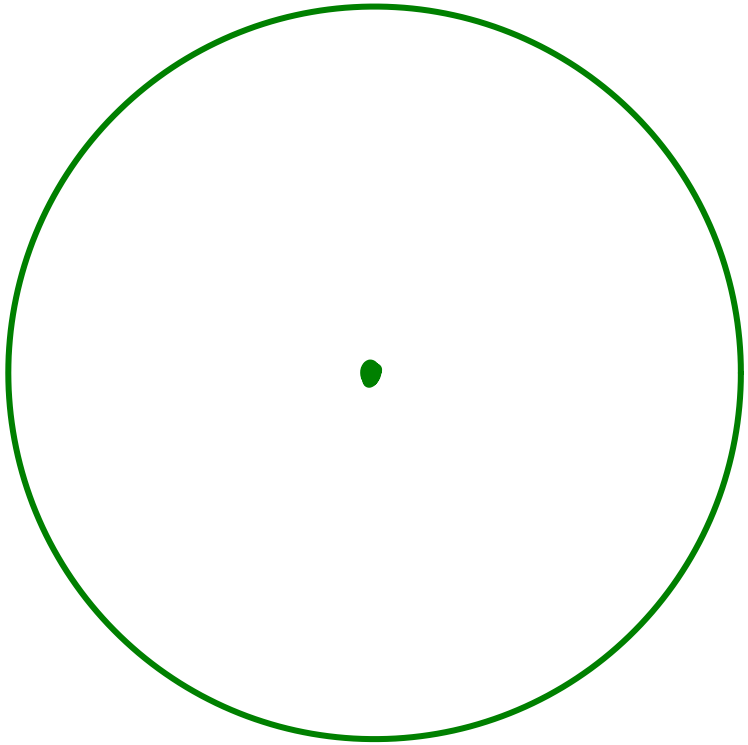

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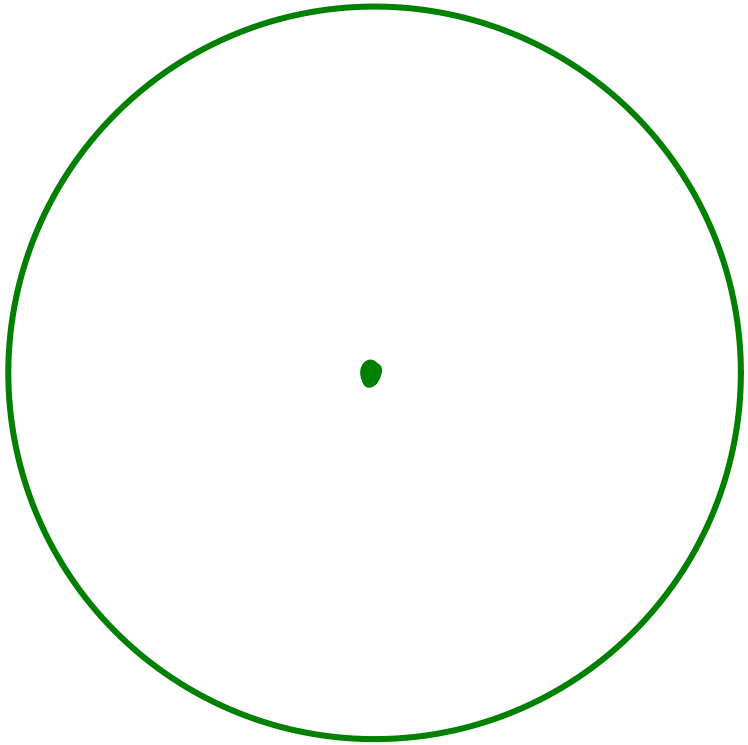

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formal fundamental solution:

$$\Phi = F(z)\, z^{\Lambda}\, e^{Q}$$

$F(z) \in G((z))$

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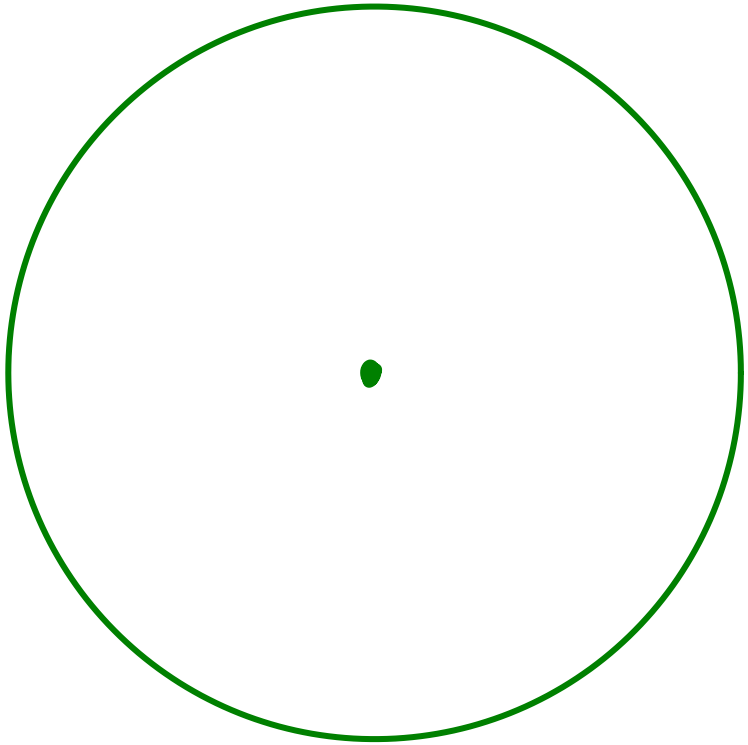

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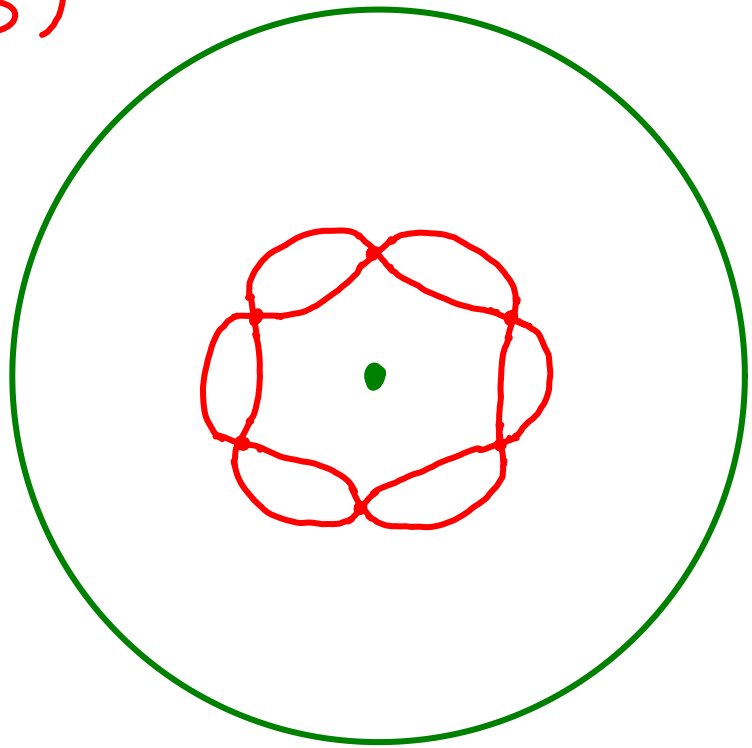

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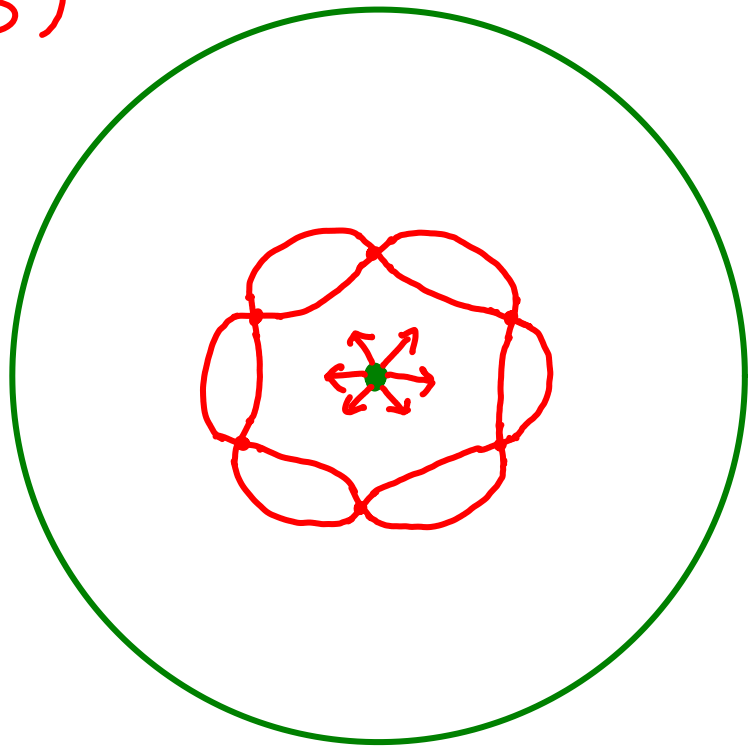

- singular directions $/A$

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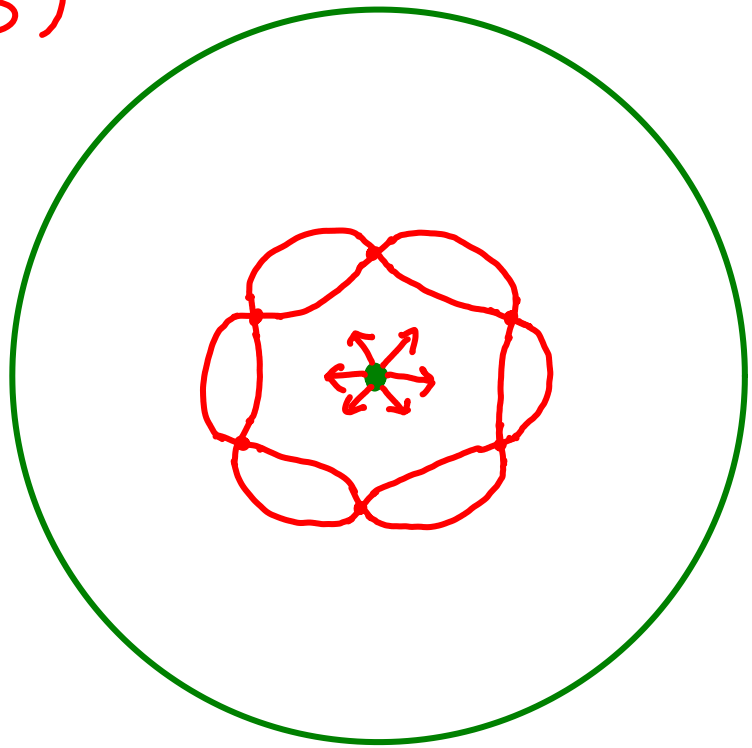

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$\exists$ "sum" F_i of F on each sector, i.e. isomorphism $\nabla_0 = dQ + \Lambda \frac{dz}{z} \cong \nabla$

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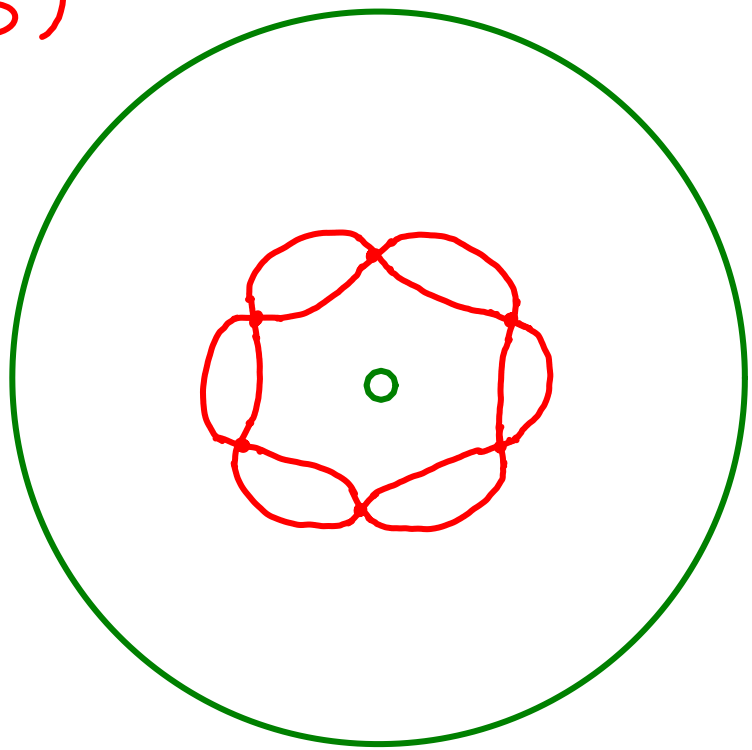

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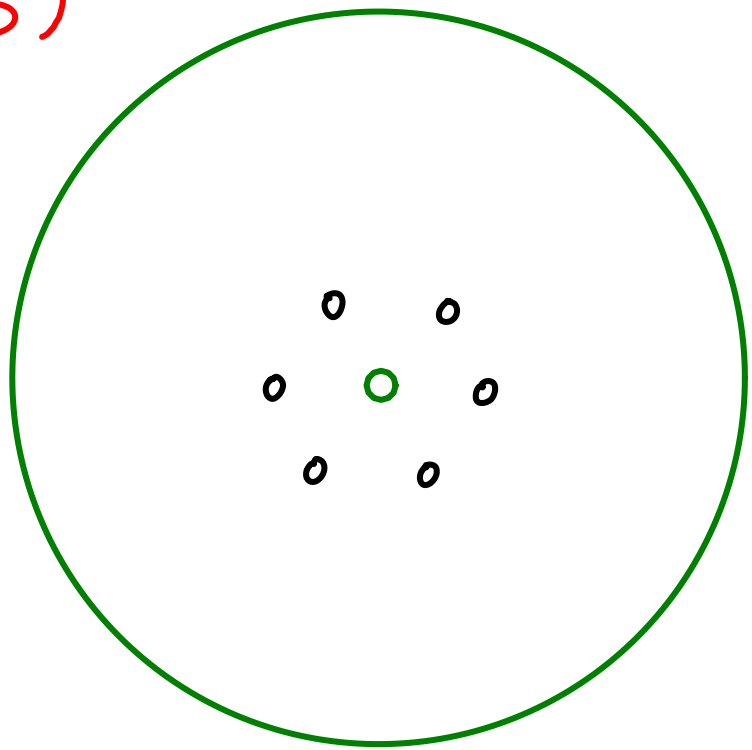

$$\nabla = dQ + \Lambda \frac{dz}{z} + \text{holom.}$$

- extra punctures $e(d)$ $\forall\, d \in \Lambda$

- singular directions Λ

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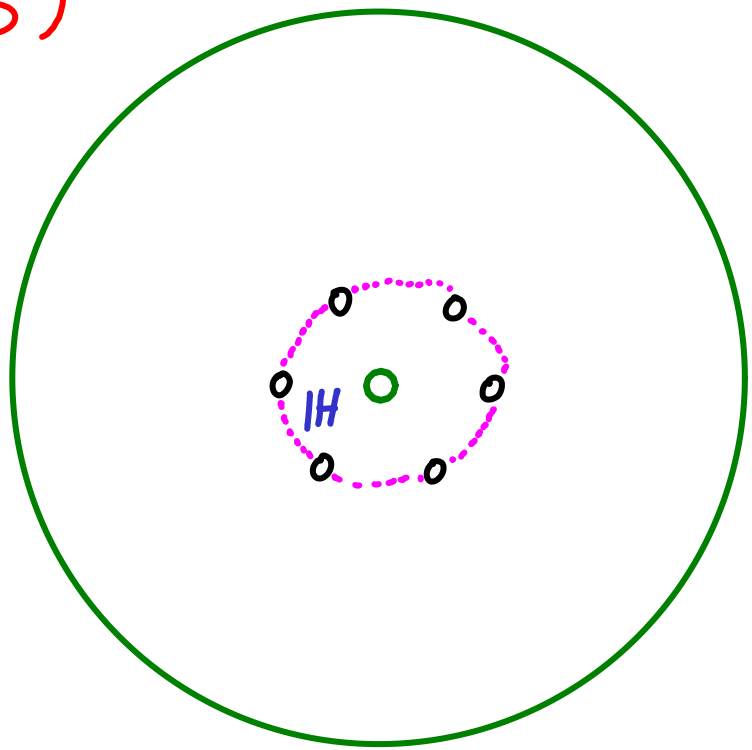

$$\nabla = dQ + \Lambda \frac{dz}{z} + \text{holom.}$$

- extra punctures $e(d)$ $\forall\, d \in A$
- halo IH

- singular directions A

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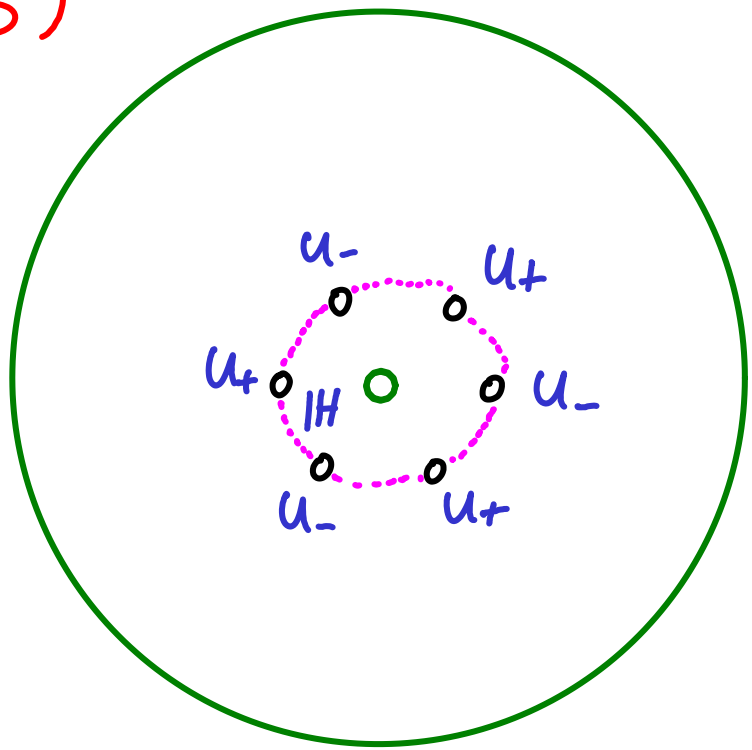

$\nabla = dQ + \Lambda \frac{dz}{z} + \text{holom.}$

- extra punctures $e(d)$ $\forall\, d \in \mathcal{A}$
- halo $\mathbb{H}$
- Stokes groups $Stok_d \subset G$ $\forall\, d \in \mathcal{A}$

$\cong U_+$ or U_- here

$\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$

- singular directions $\mathcal{A}$

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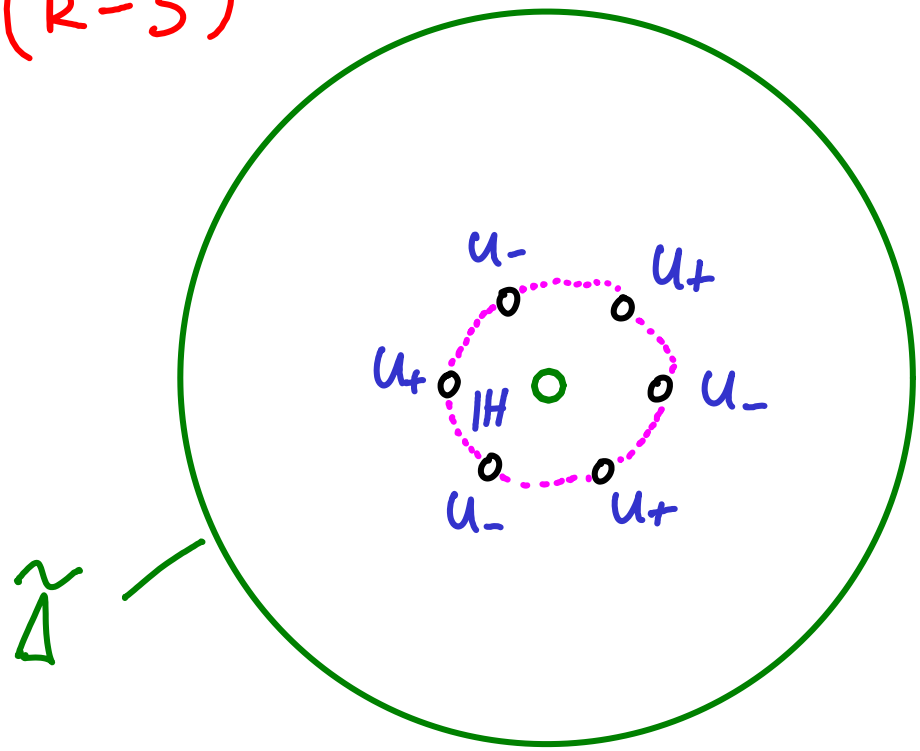

$\tilde{\Delta}$

$\nabla = dQ + \Lambda \frac{dz}{z} + \text{holom.}$

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Fix G (e.g. $GL_n(\mathbb{C})$)

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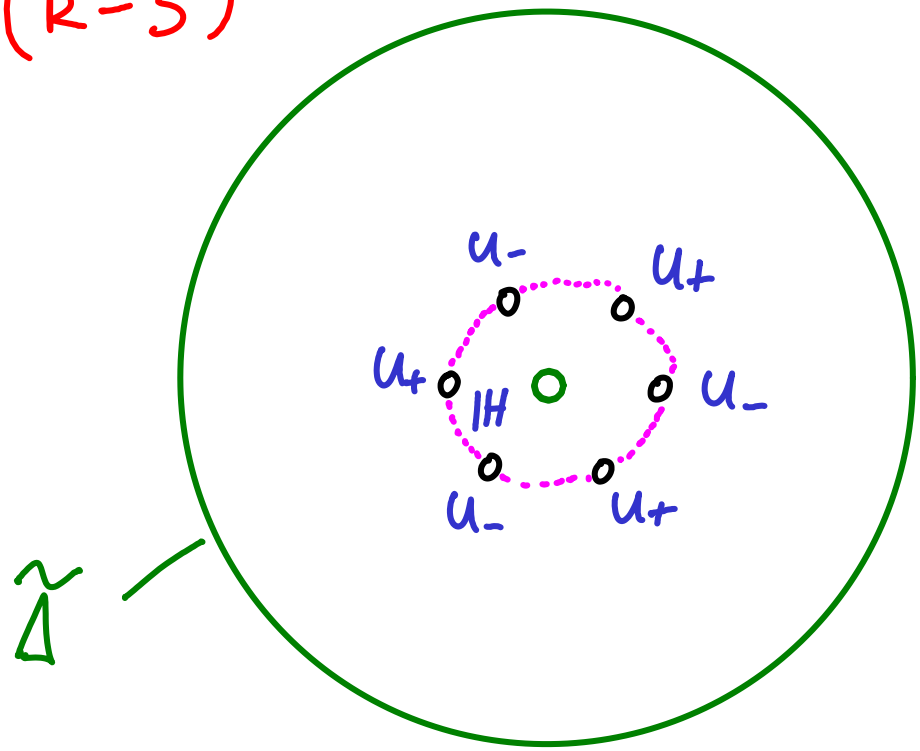

$\tilde{\Delta}$

$\nabla = dQ + \Lambda \frac{dz}{z} + \text{holom.}$

Stokes local system:

- singular directions $\mathcal{A}$

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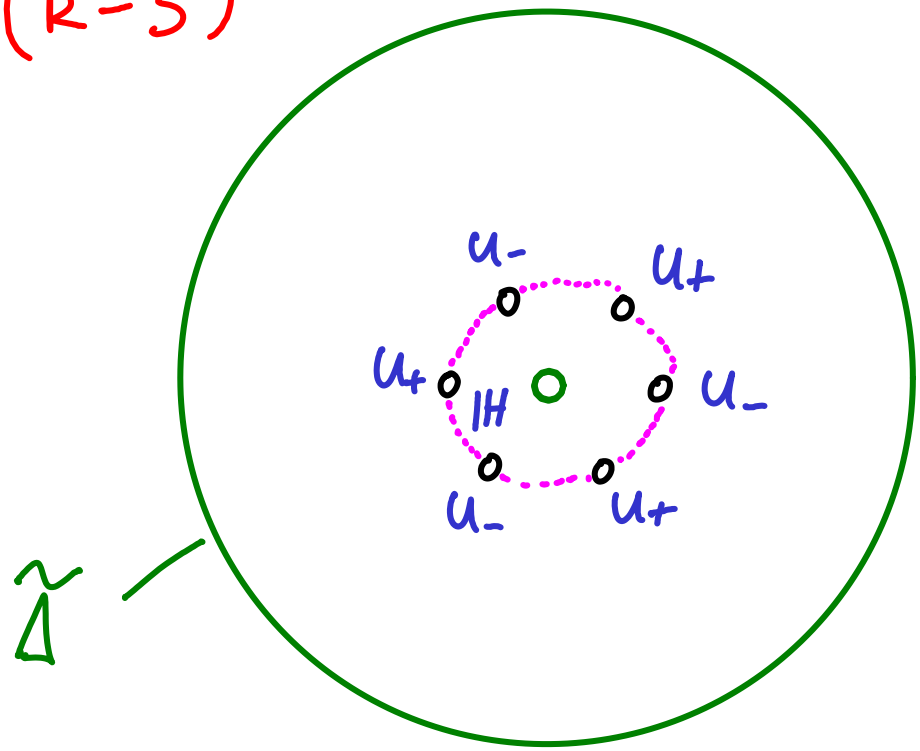

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Stokes local system:

G-local system on $\tilde{\Delta}$

- flat reduction to $H = C_G(Q)$ in IH
- monodromy around $e(d)$ in St_d $\forall d \in A$

- singular directions A

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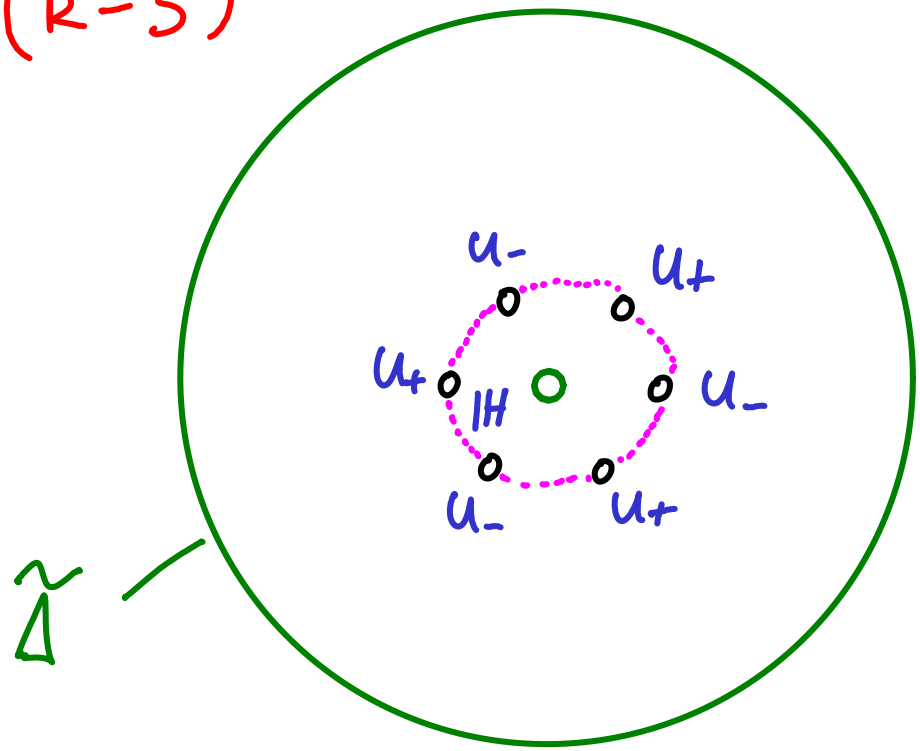

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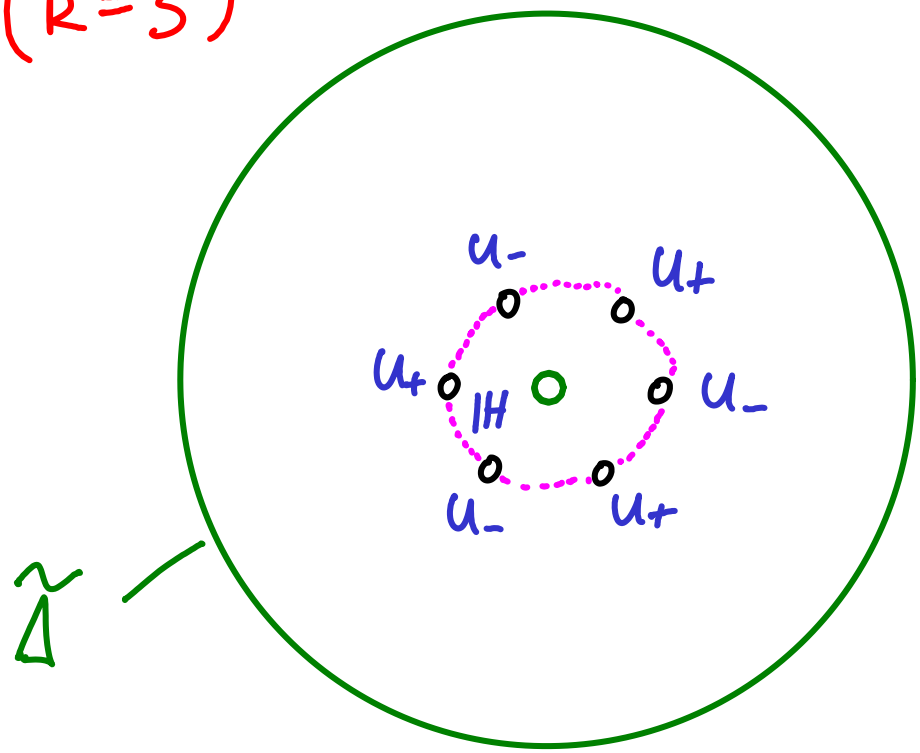

Classify ?

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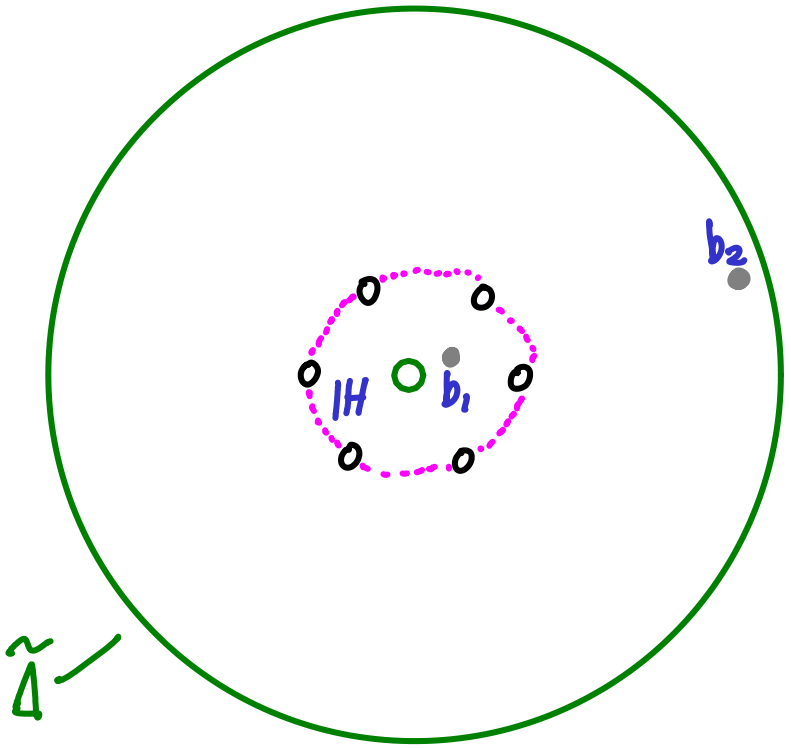

basepoints b_1, b_2

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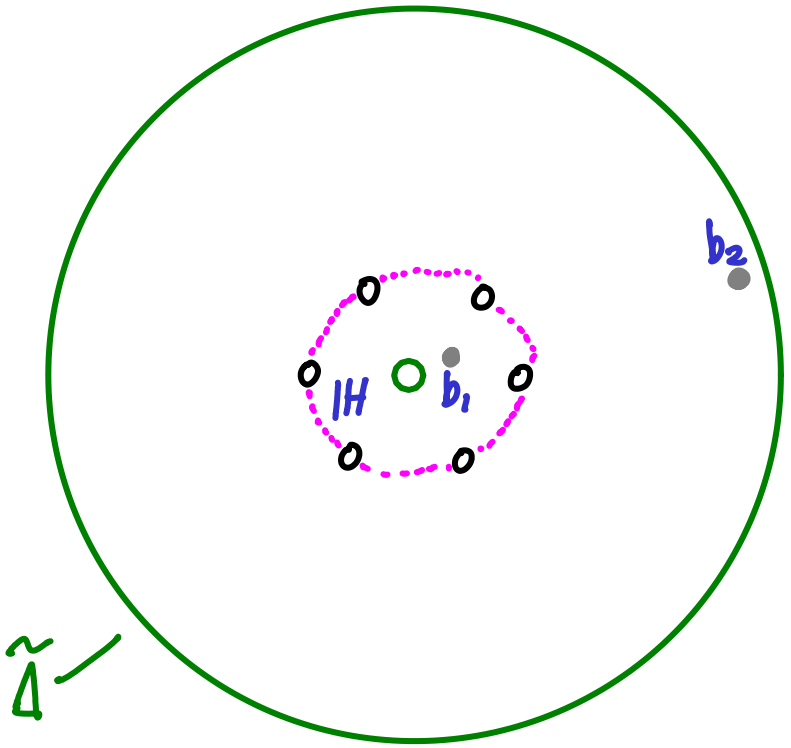

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$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$

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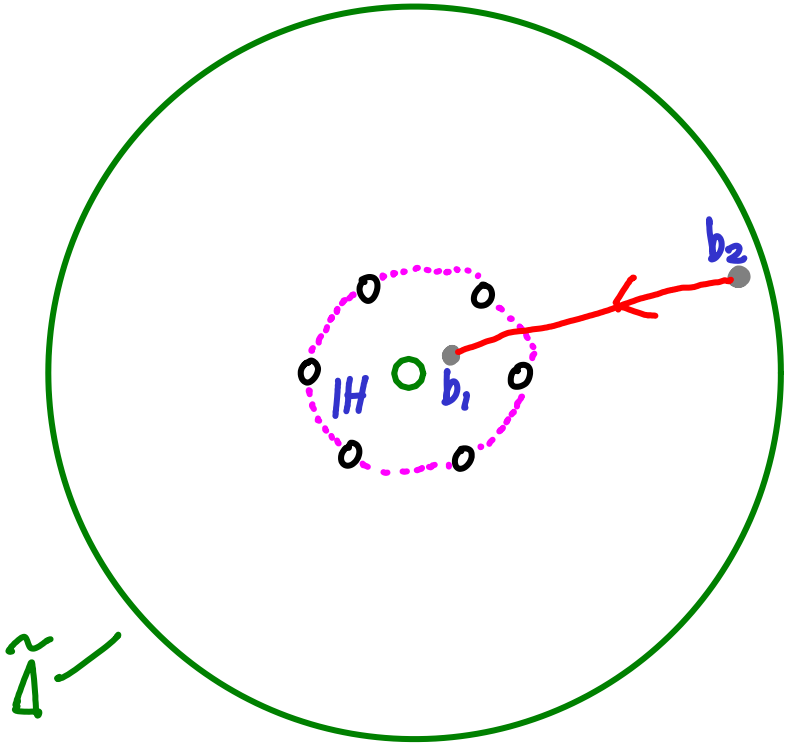

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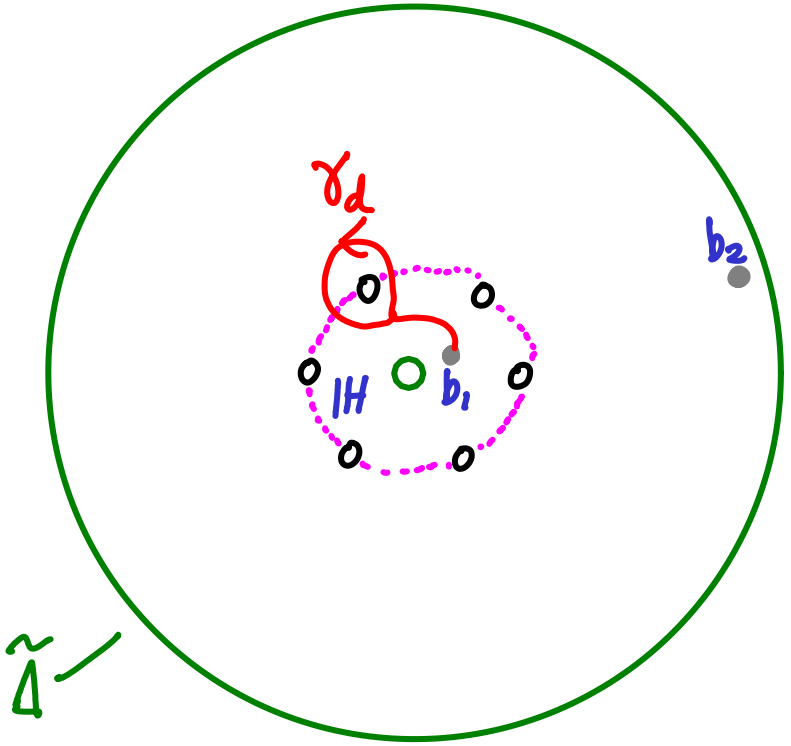

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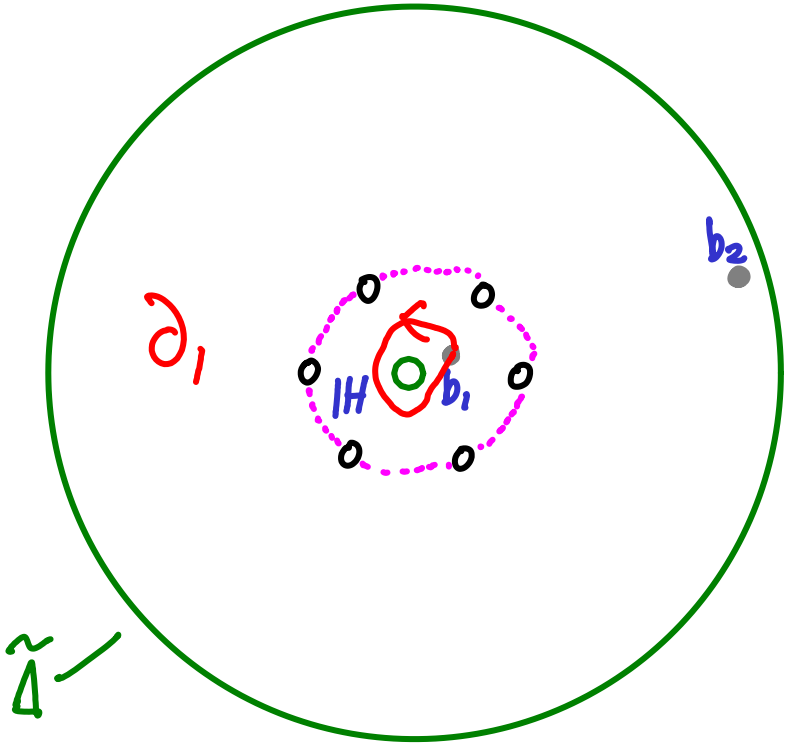

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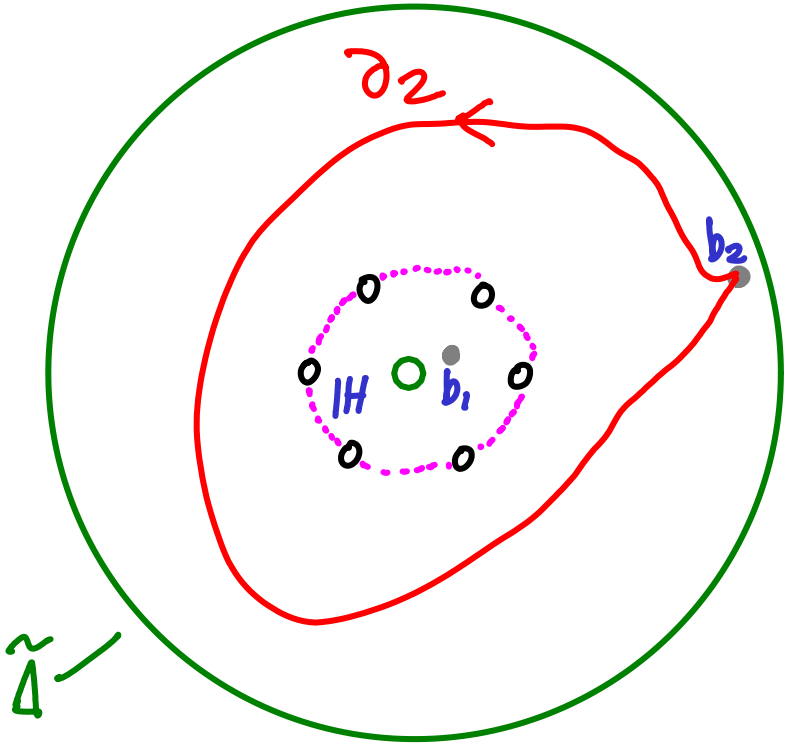

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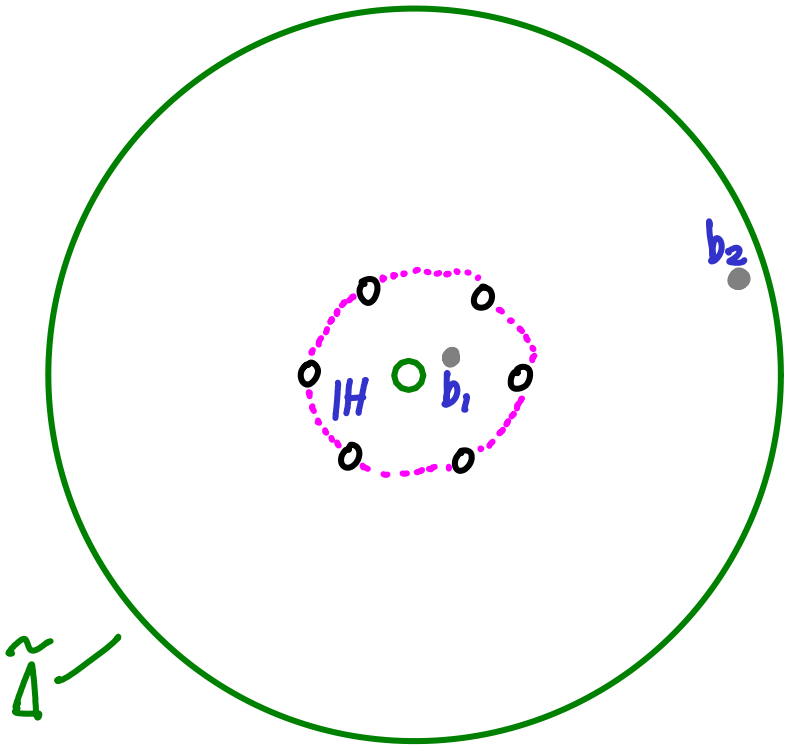

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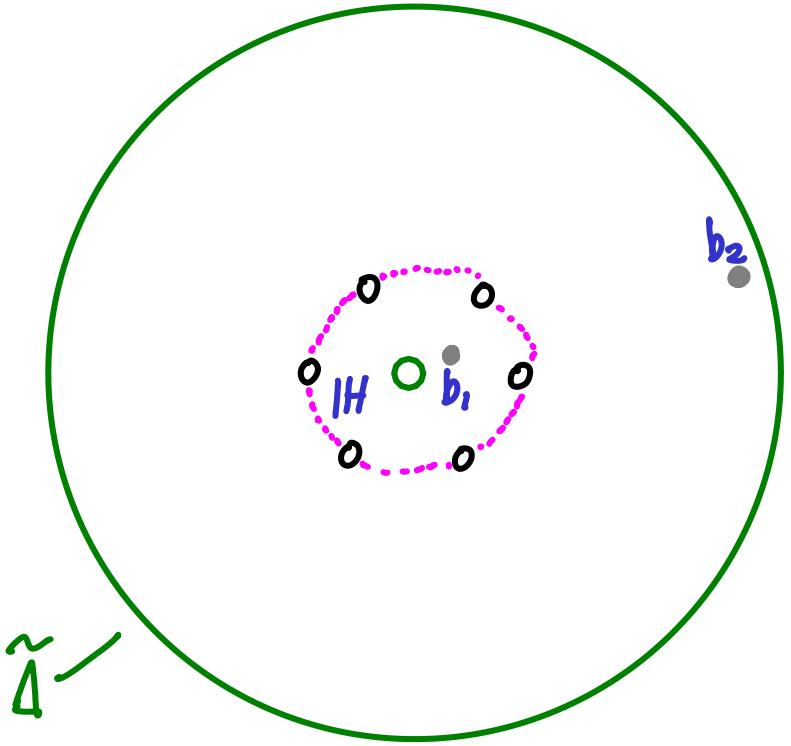

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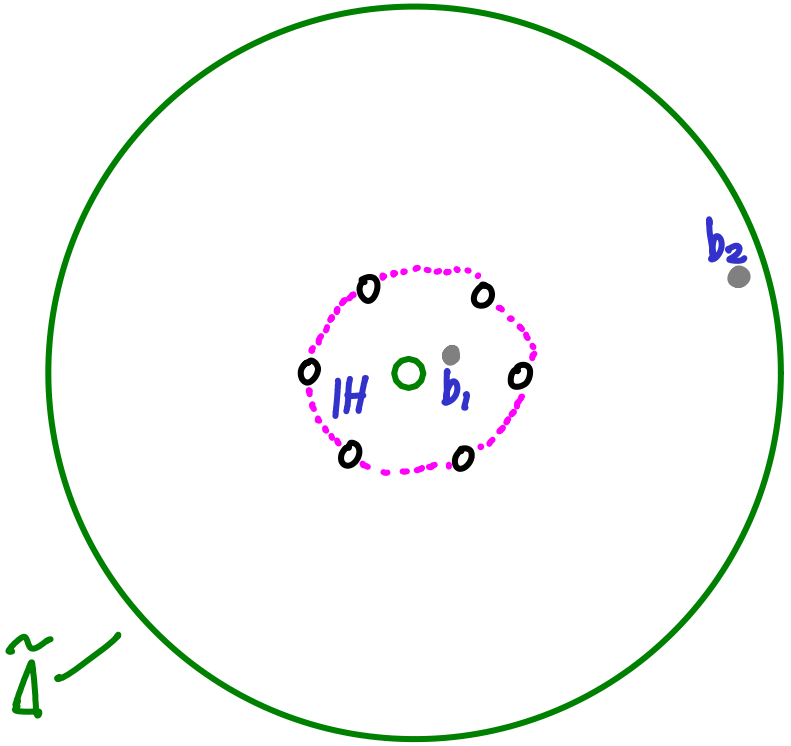

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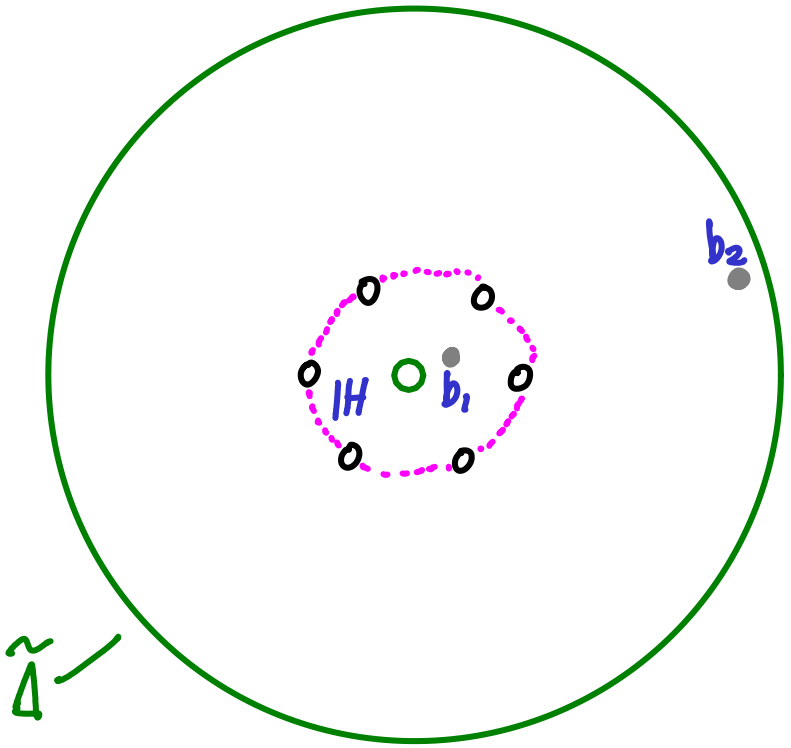

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Lemma $\left(\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_r & 1 \end{pmatrix} \right)_{11} = (a_1, b_1, \ldots, a_r, b_r)$

— Euler's continuants are group valued moment maps

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[Similarly for (2009-2015) $V = V_1 \oplus V_2$ any dimension; Γ any "fission graph"]

$$\mu(a_1, \ldots, b_{k-1}) = \left((a_1, b_1, \ldots, a_{k-1}, b_{k-1}), (b_{k-1}, \ldots, b_1, a_1)^{-1}\right)$$

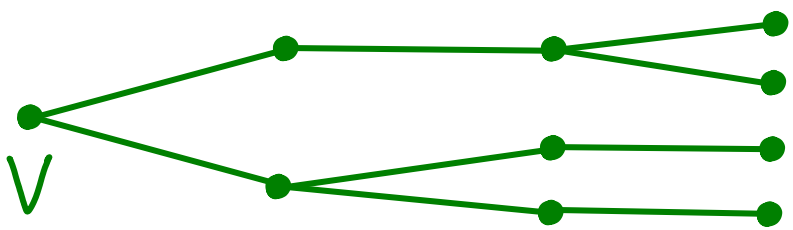
Fission graphs (arXiv 0806, apx C)

$G = GL(V)$

$$Q = A_r/z^r + \cdots + A_1/z \qquad (A_i \in t)$$

$$= A_r w^r + \cdots + A_1 w \qquad w = 1/z$$

$r = 3$:



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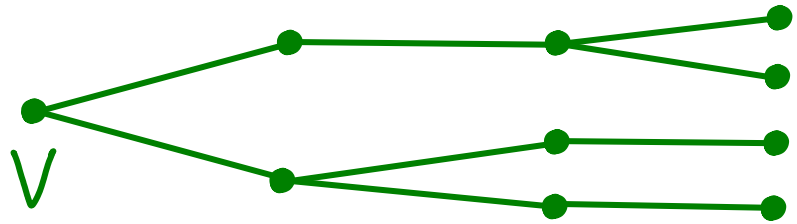

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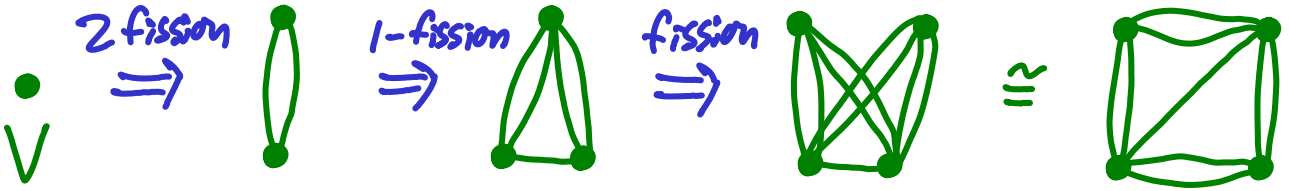
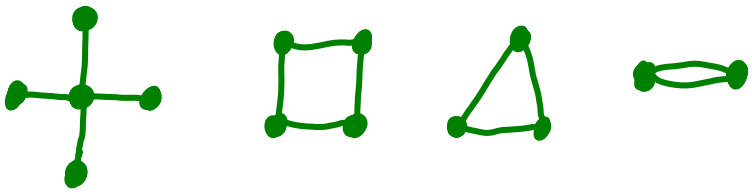


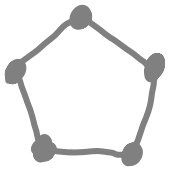
$\simeq$

"fission graph" $\Gamma(Q)$

- $r = 2$ get all complete k-partite graphs
- e.g.



but not



$Q = \mathrm{diag}(q_1, \ldots, q_n) \Rightarrow$ nodes $= \{1, \ldots, n\}$, # edges $i \leftrightarrow j = \deg_w(q_i - q_j) - 1$

Wild Character Varieties

In this example $\left((\mathbb{P}^1, 0, Q) \quad Q = A/\mathbb{Z}^k, \quad GL_2(\mathbb{C}) \right)$

$$\mathcal{M}_B = \tilde{\mathcal{M}}_B \,/\!\!/_{(q_1,q_2)}\, H$$

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$\Gamma =$, $V = \mathbb{C} \oplus \mathbb{C}$

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E.g. $k=3$ (Painlevé 2 Betti space)

$$\mathcal{M}_B \cong \left\{ xyz + x + y + z = b - b^{-1} \right\} \quad b \in \mathbb{C}^* \text{ constant}$$

(Flaschka-Newell surface)

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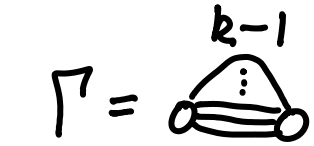

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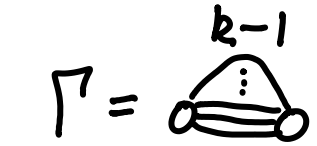

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$$\mathcal{M}^* \xrightarrow{\text{RHB}} \mathcal{M}_B$$

$$\| \qquad\qquad \|$$

$$\mathrm{Rep}(\Gamma, V) /\!\!/_\lambda H \qquad \mathrm{Rep}^*(\Gamma, V) /\!\!/_q H$$

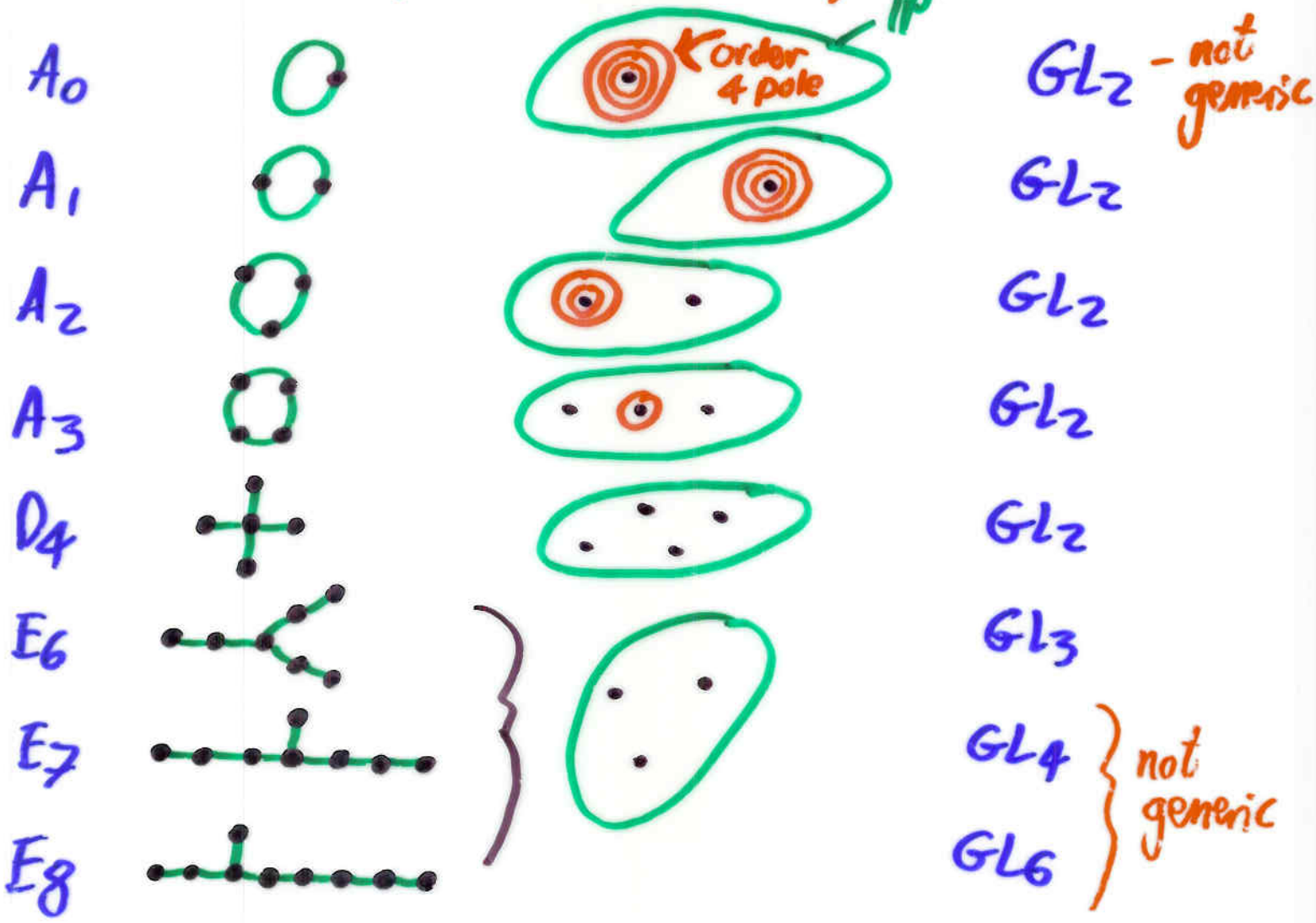
Examples

"Quaternionic Curves" = { $\dim_{\mathbb{H}} = 1$; real 4 mfds

- simplest examples have open pieces

≅ Kronheimer's ALE spaces

Cx. sympl. not isometrically

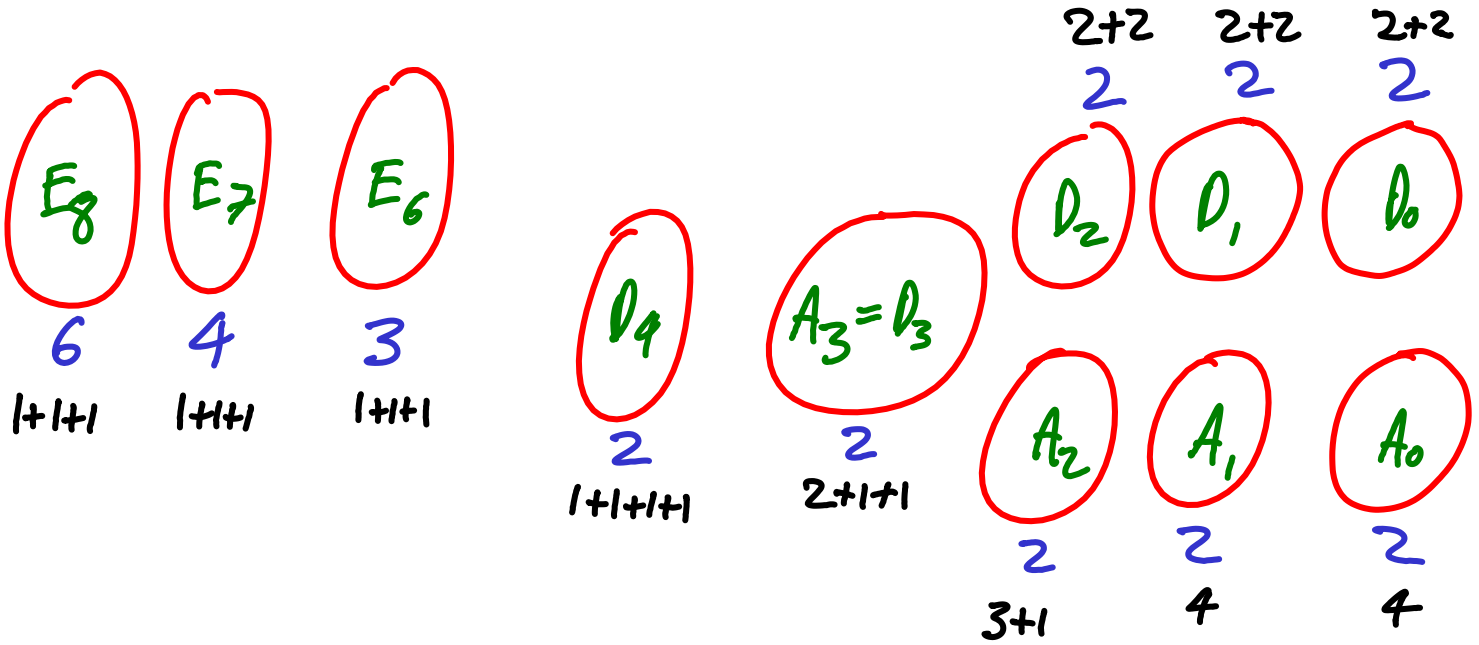


cf. Cherkis-Kapustin, ..., Painlevé...

[Slide from talk at IAS Princeton, November 2007]

Conjectural classification (of $\mathcal{U}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607)



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minimal rank of bundles

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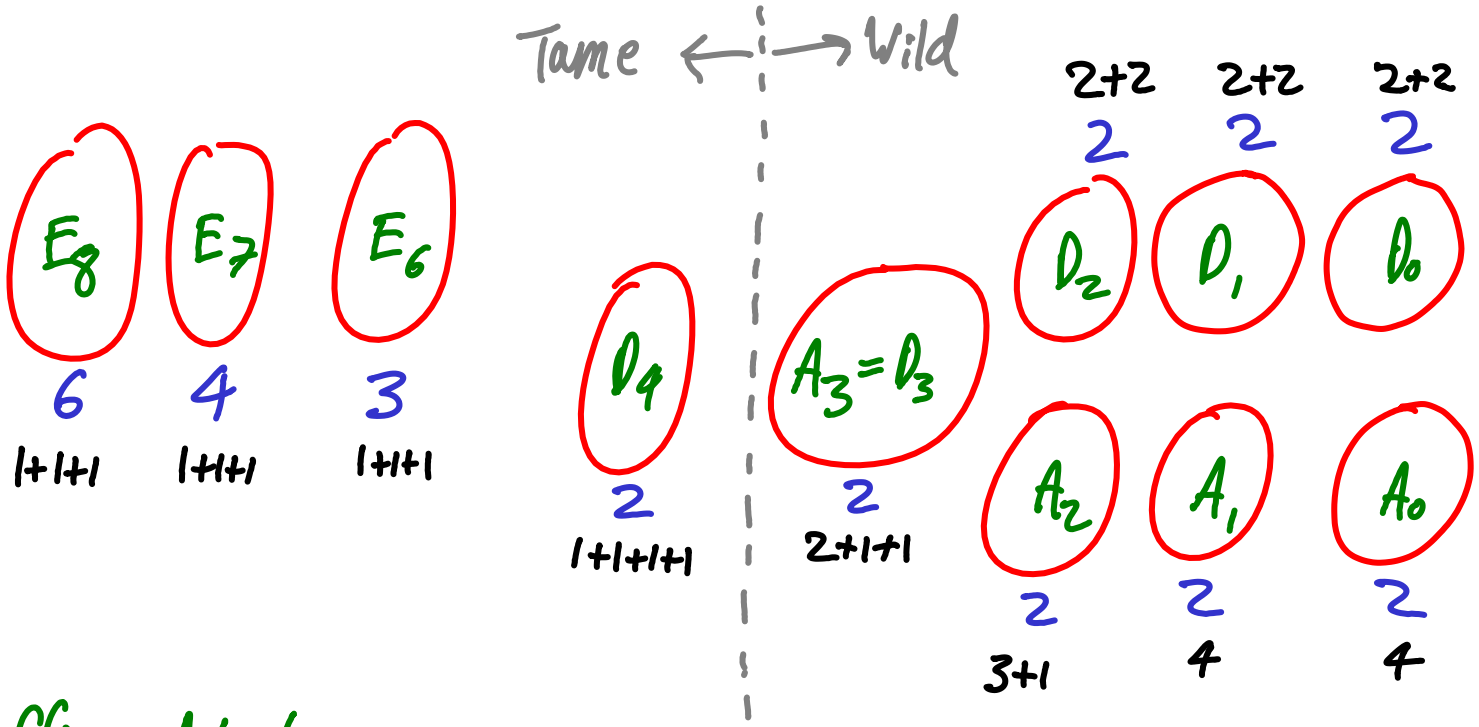

affine Weyl group

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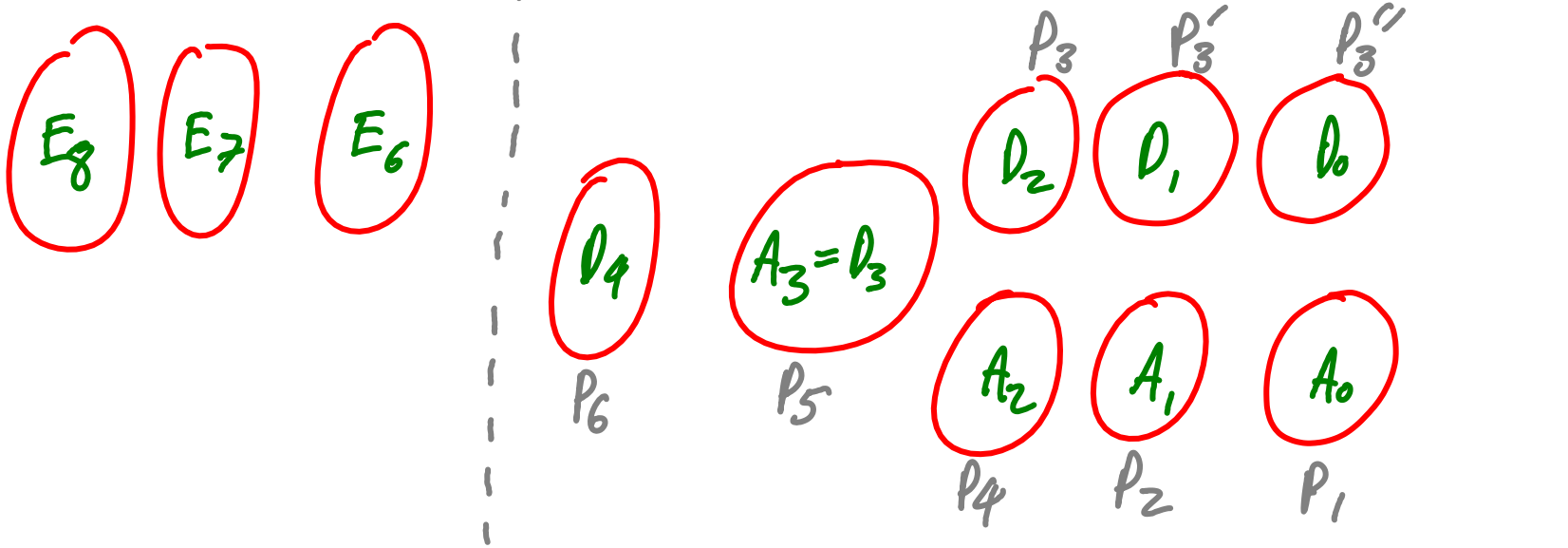
pole orders

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607)



Phase spaces for Painlevé differential equations

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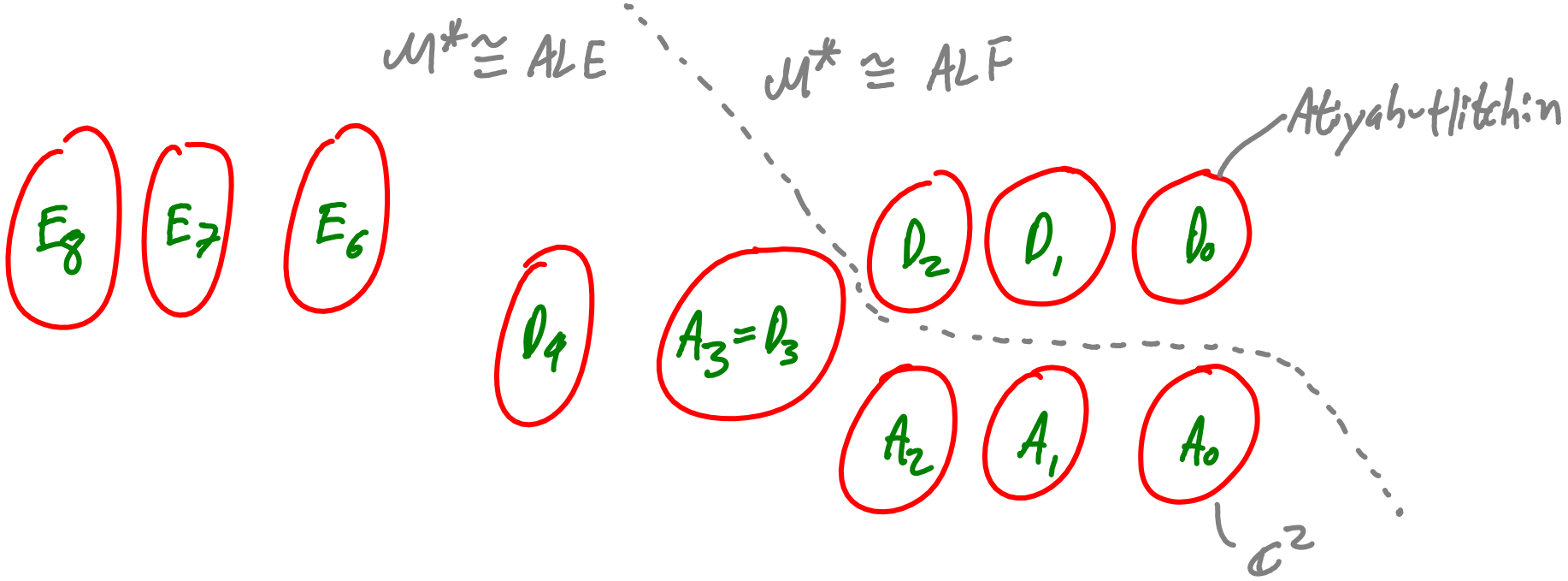
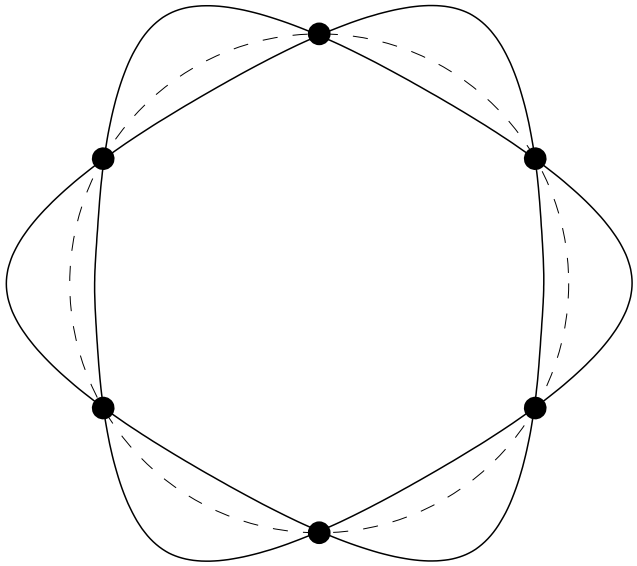


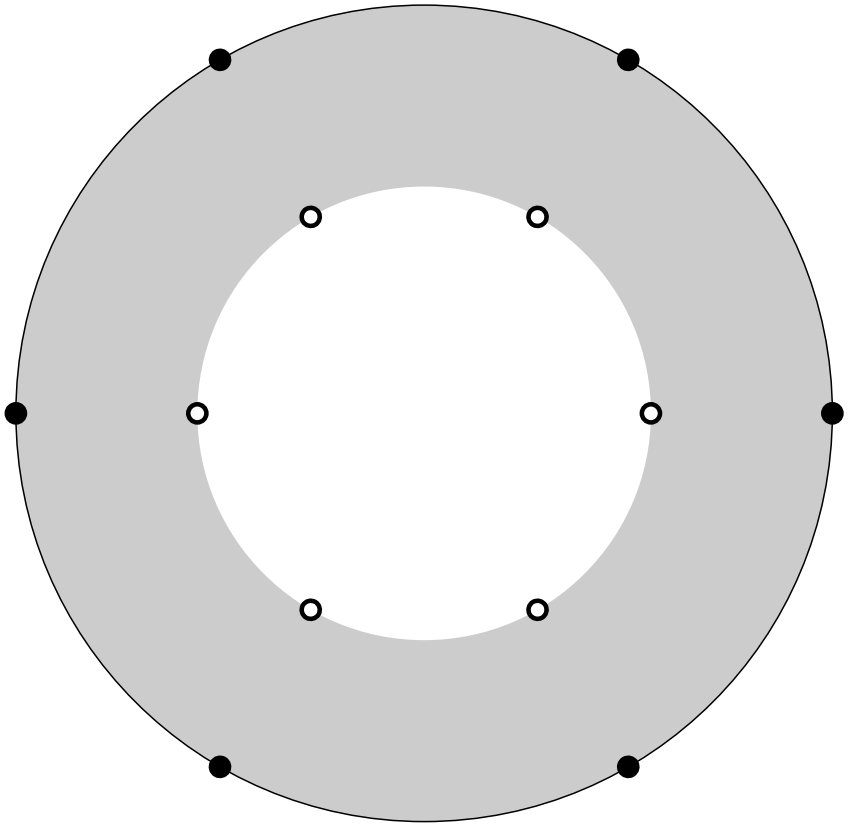
[$\mathcal{M}^* \subset \mathcal{M}$ open piece where bundle holom. trivial]

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



Stokes diagram with Stokes directions



Halo at ∞ with singular directions

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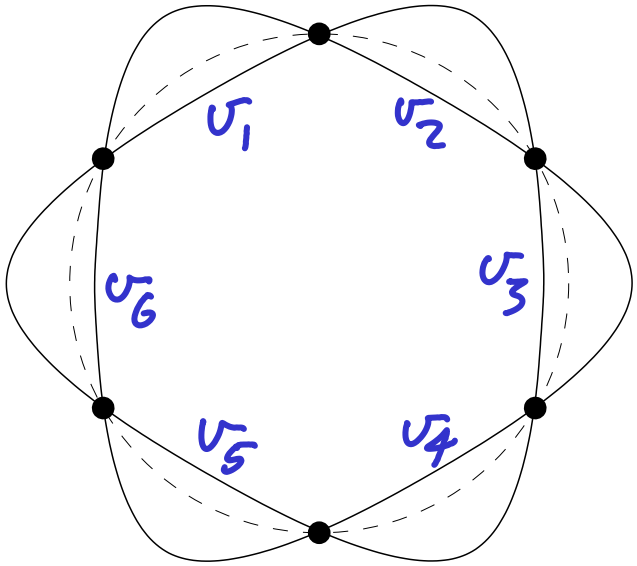

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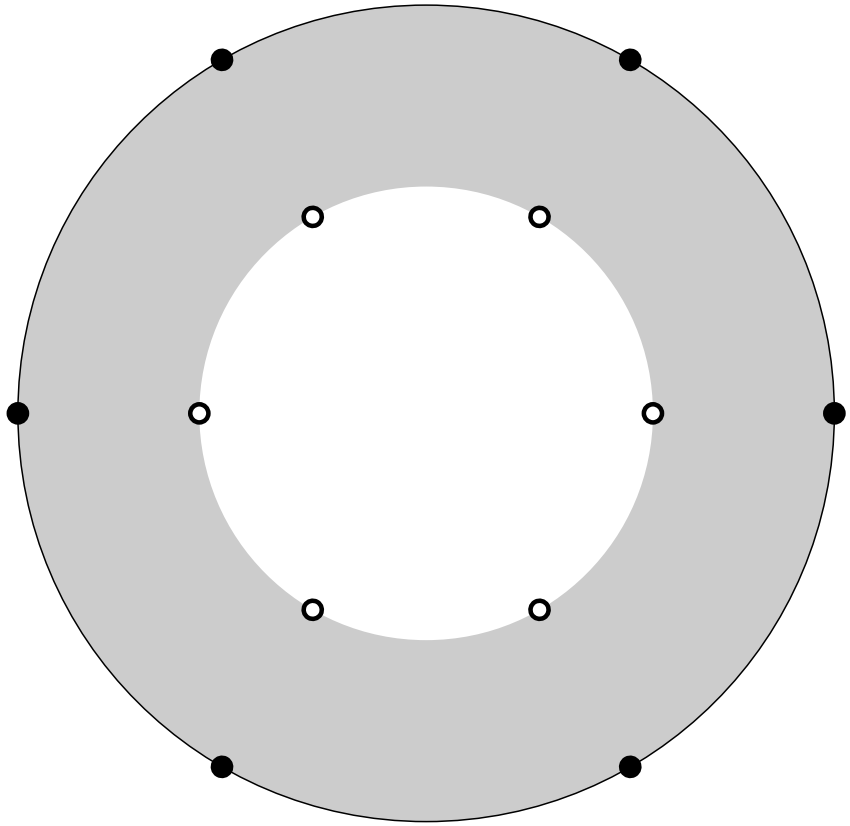

Halo at ∞ with singular directions

Subdominant solutions $\quad V_i \not\parallel V_{i+1}$

Stokes structures

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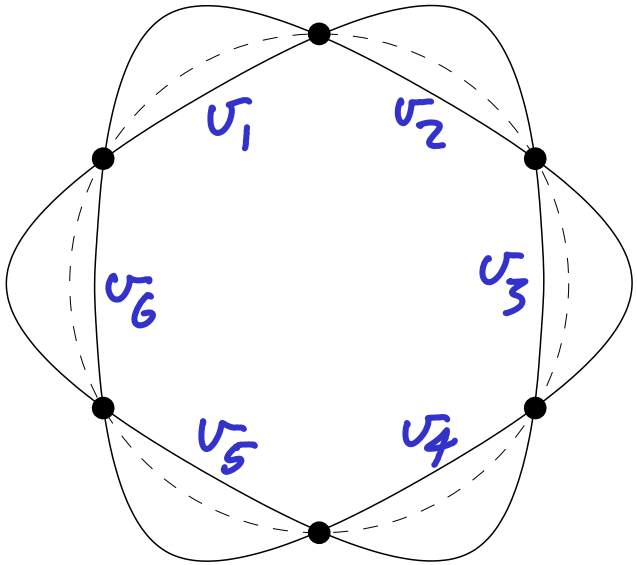

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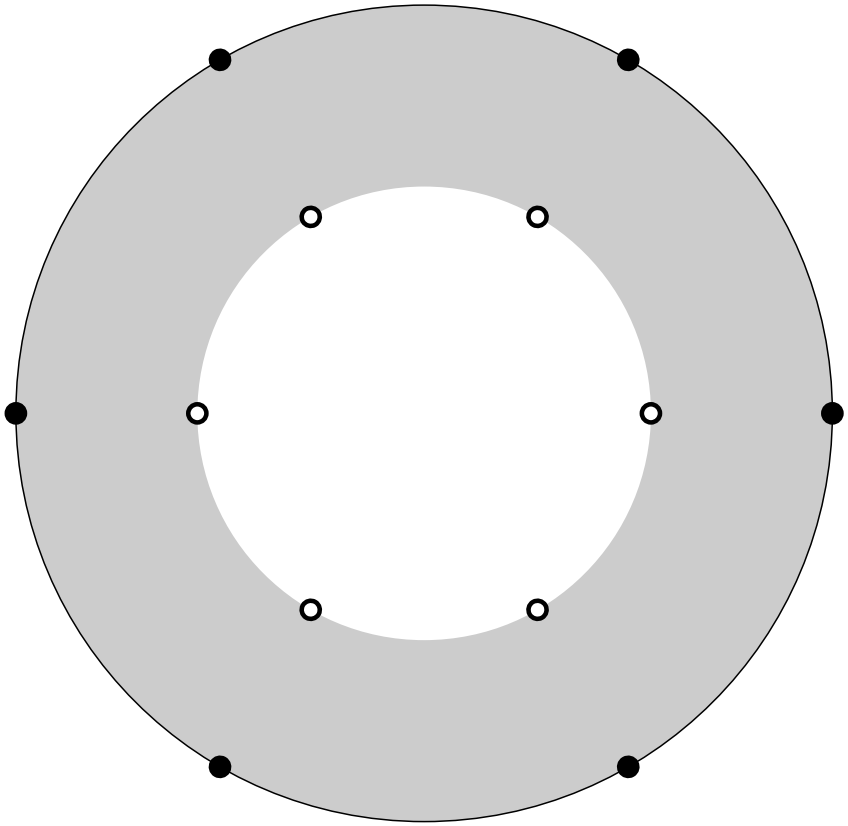


Halo at ∞ with singular directions

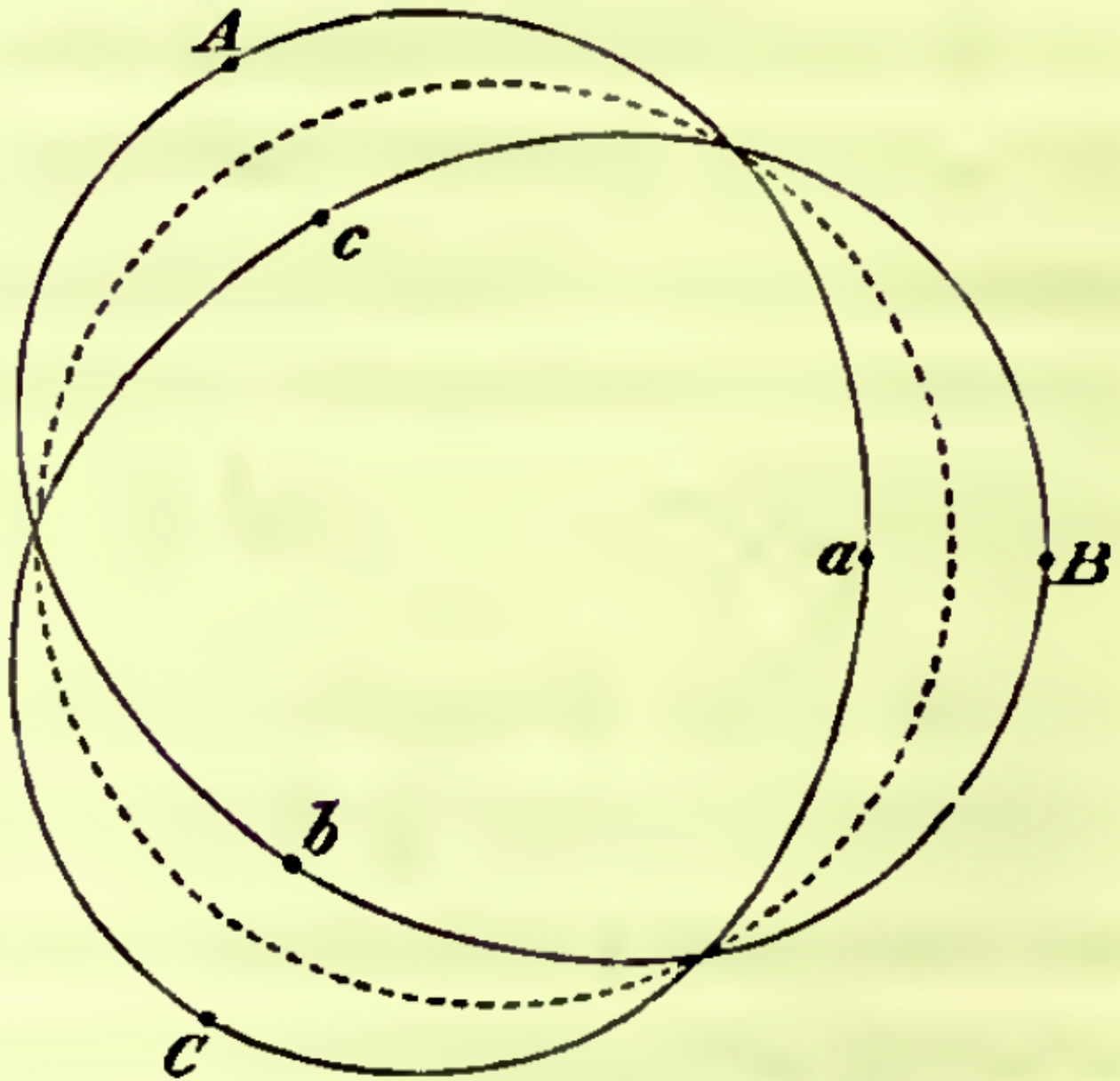
Subdominant solutions $v_i \not\parallel v_{i+1}$

$$\mathcal{M}_B \cong \{ xyz + x + y + z = b - b^{-1} \}$$

$$\cong \left\{ (p_1, \dots, p_6) \in (\mathbb{P}^1)^6 \;\middle|\; \begin{array}{c} p_i \neq p_{i+1} \mod 6 \\ \dfrac{(p_1-p_2)(p_3-p_4)(p_5-p_6)}{(p_2-p_3)(p_4-p_5)(p_6-p_1)} = b^2 \end{array} \right\} / PSL_2(\mathbb{C})$$

other the *inferior*
the existence and
different values of
ng a radius vector
e angle θ take two
and inwards from
al to the real part
perior and inferior
or in other words
nvenience suppose
d with the radius.

Fig. 1.



The curve will evidently have the form represented

§3 Odd continuants (work with D. Yamakawa)

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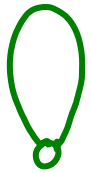
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$

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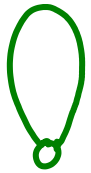
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$, $V = \mathbb{C}^d$

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(PB 2008, unpublished)

- So get Calogero-Moser spaces, ADHM spaces as $\mathcal{M}^*$
- $\mathcal{M}^+ = \mathbb{C}^2$ for Painlevé 1 $(d=1)$

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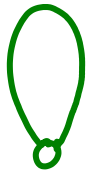
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $V = \mathbb{C}^d$ $A_0^{(1)}$

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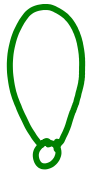
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$, $V = \mathbb{C}^d$

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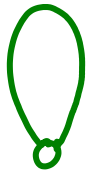
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$, $V = \mathbb{C}^d$
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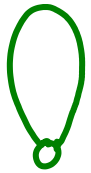
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $V = \mathbb{C}^d$ $A_0^{(1)}$

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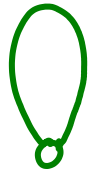
$\Gamma =$  $V = \mathbb{C}^d$

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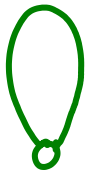
§3 **Odd continuants** (work with D. Yamakawa) $\Gamma =$  $V = \mathbb{C}^d$

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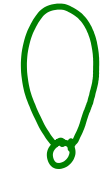
Other reductions:

$$\mathrm{Rep}^*\left(\begin{array}{c} d \\ \bigcirc \\ 1 \end{array} , \mathbb{C}^d \oplus \mathbb{C} \right) /\!\!/_q H \cong \mathcal{M}_B\left(hP_1^{(d)}\right) \quad \dim 2d$$

higher/hyperbolic/Hilbert

Painlevé 1

§3 Odd continuants (work with D. Yamakawa) $\Gamma =$  $V = \mathbb{C}^d$

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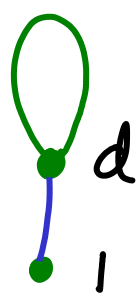
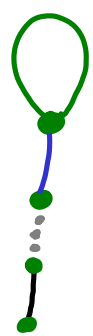
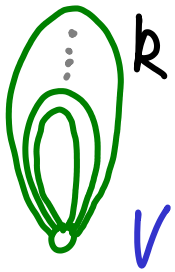
$\mathrm{Rep}^*\Big($  $, \mathbb{C}^d \oplus \mathbb{C} \Big)_{/\!/_q H} \cong \mathcal{M}_B(hP_1^{(d)})$ dim $2d$

higher/hyperbolic/Hilbert Painlevé 1

$\mathrm{Rep}^*\Big($  $, \oplus V_i \Big)_{/\!/_q H} \cong \mathcal{M}_B(\text{matrix } P_1)$

§3 Odd continuants (work with D. Yamakawa)

More generally if $\Gamma =$ 

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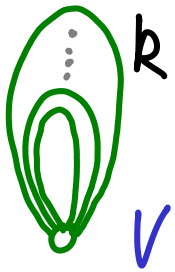
More generally if $\Gamma =$ 

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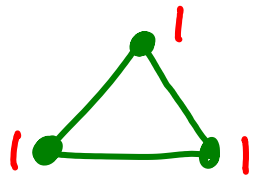
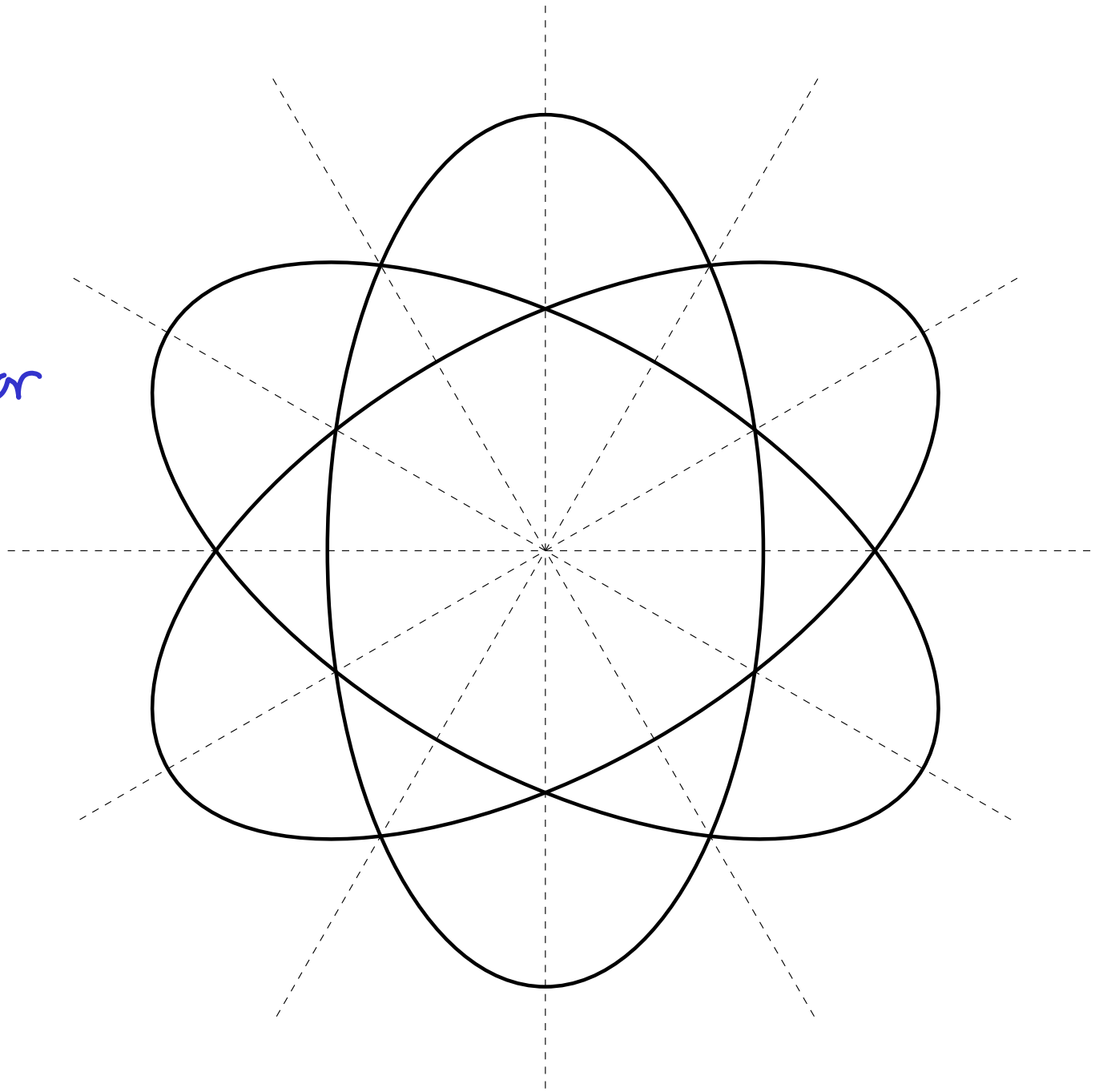
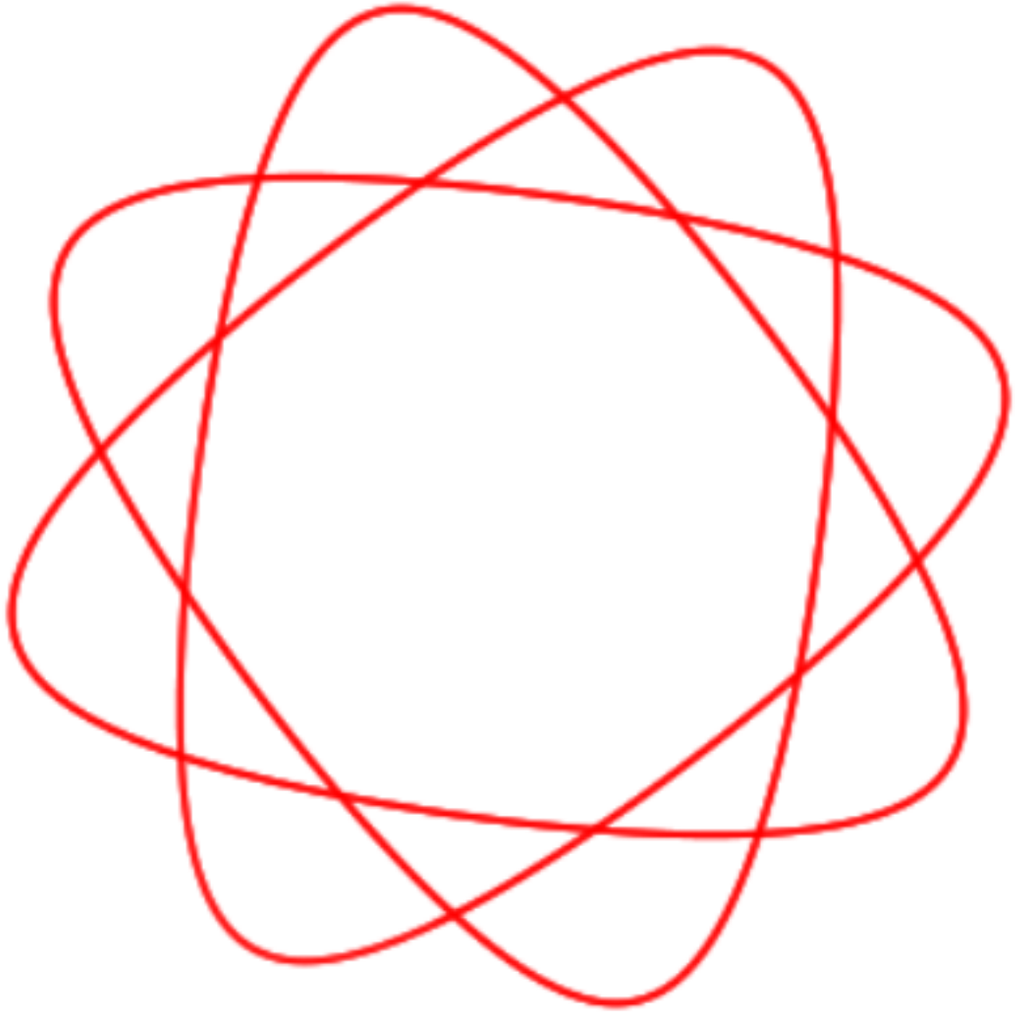
– and similarly for any twisted irregular type Q (any G)

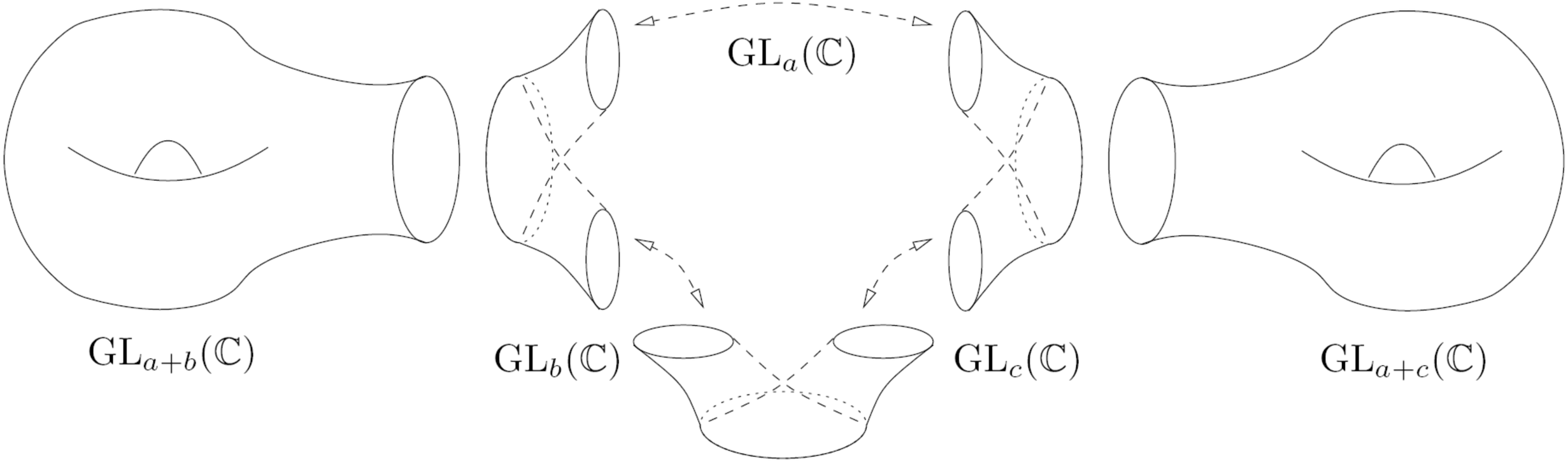
Stokes diagram for

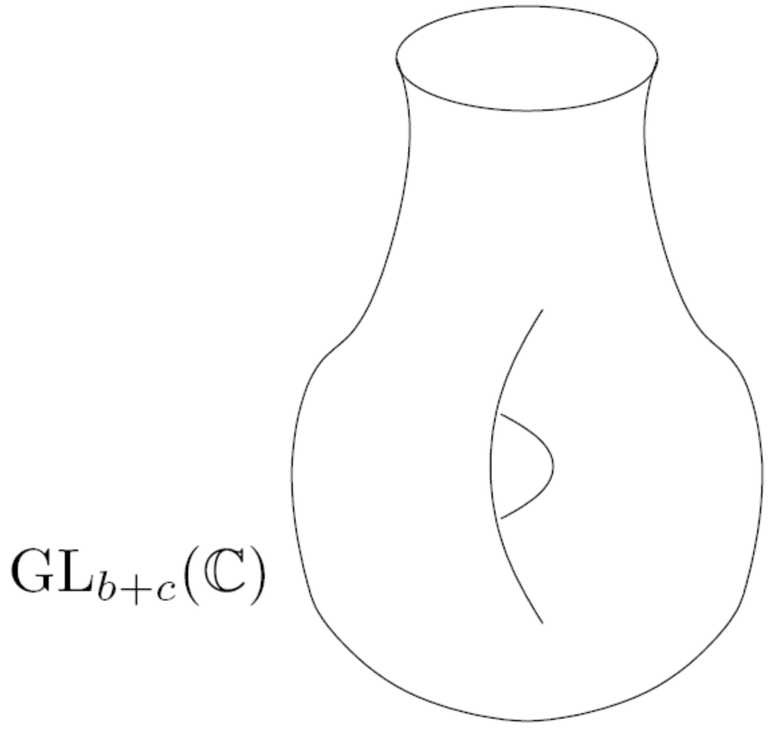


(generic reading)









- complex symplectic (An. Inst Fourier 2009)
- is it hyperkähler?