Connections on Curves
and
wild character varieties

NS@50

Chennai 2015

Phil Boalch

CNRS & Orsay

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

Σ smooth compact curve $\Longrightarrow$ $\mathcal{M}(\Sigma)$ Hyperkahler manifold

$\mathcal{M}(\Sigma) \overset{\text{Hitchin-Simpson}}{=} \mathcal{M}_{Dol}(\Sigma)$

$\mathcal{M}(\Sigma) \underset{\text{Corlette-Donaldson}}{=} \mathcal{M}_{DR}(\Sigma)$

$$\mathcal{M}_{Dol}(\Sigma) \overset{\text{Non-abelian Hodge}}{\cong} \mathcal{M}_{DR}(\Sigma) \overset{\text{RH isomorphism}}{\cong} \mathcal{M}_{B}(\Sigma)$$

3 algebraic structures

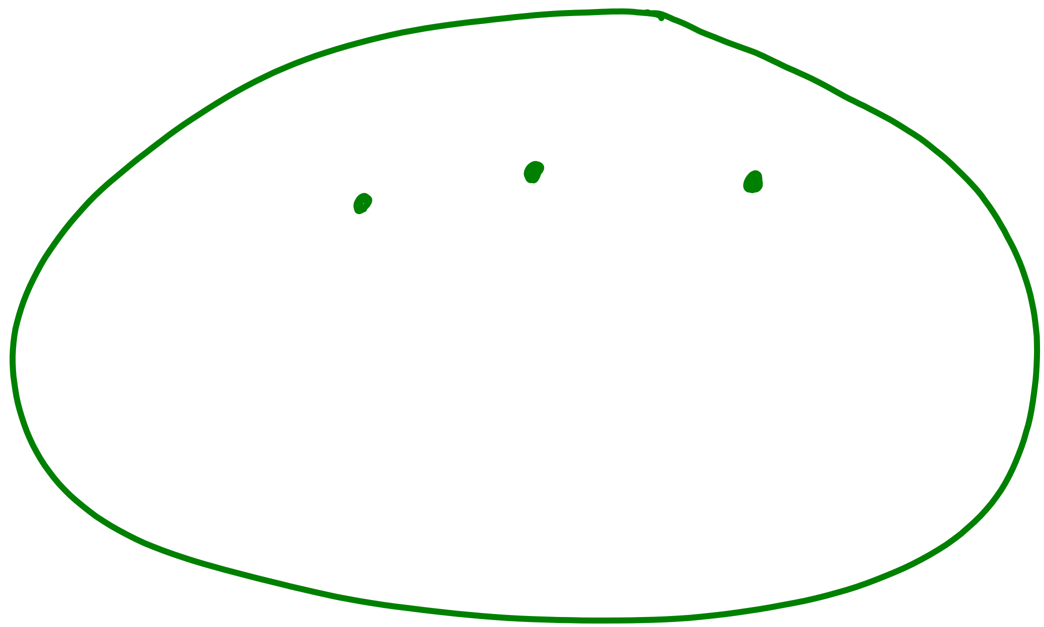
X a space, $x \in X$ a basepoint

$$\pi_1(X, x) = \left\{ \begin{array}{c} \text{homotopy classes of loops in } X \\ \text{based at } x \end{array} \right\}$$

— group under composition of loops

[$\gamma_2 \circ \gamma_1$ means go around γ_1 then γ_2]

E.g. $X = \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ (m-punctured two sphere)



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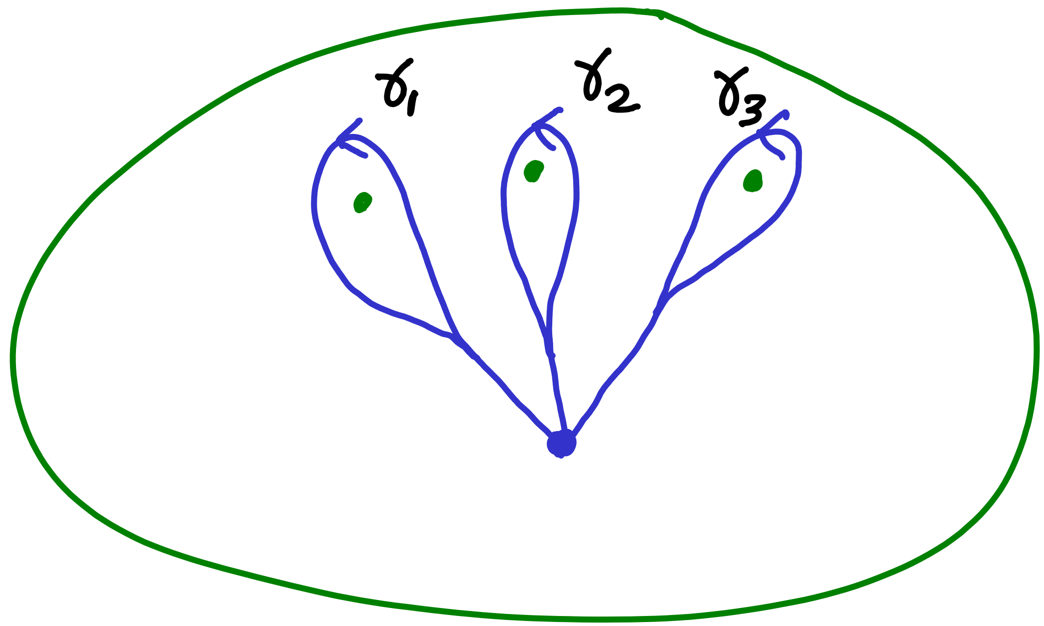

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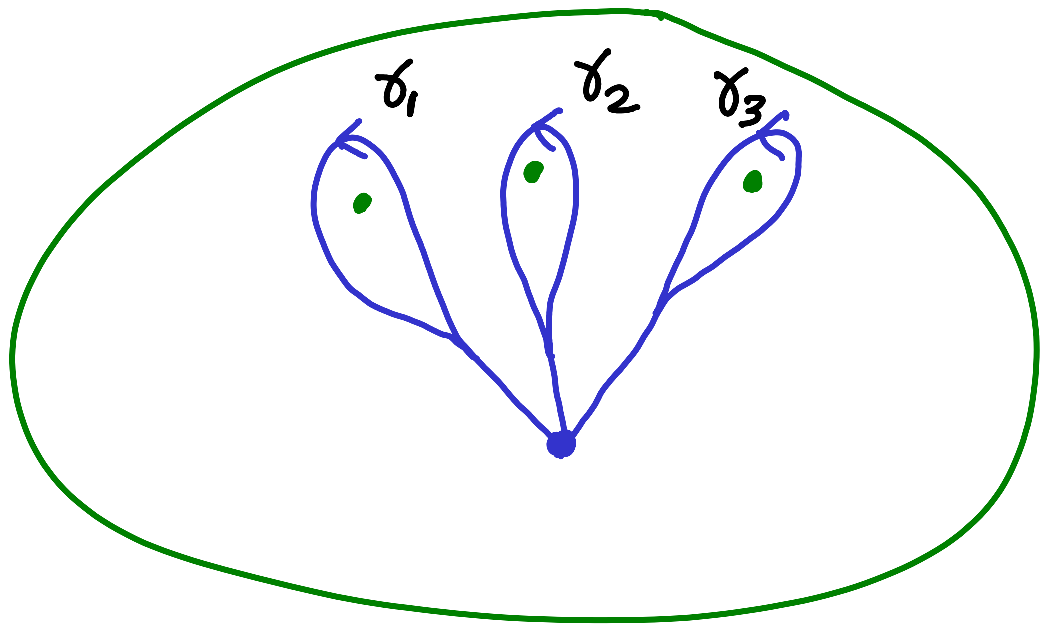
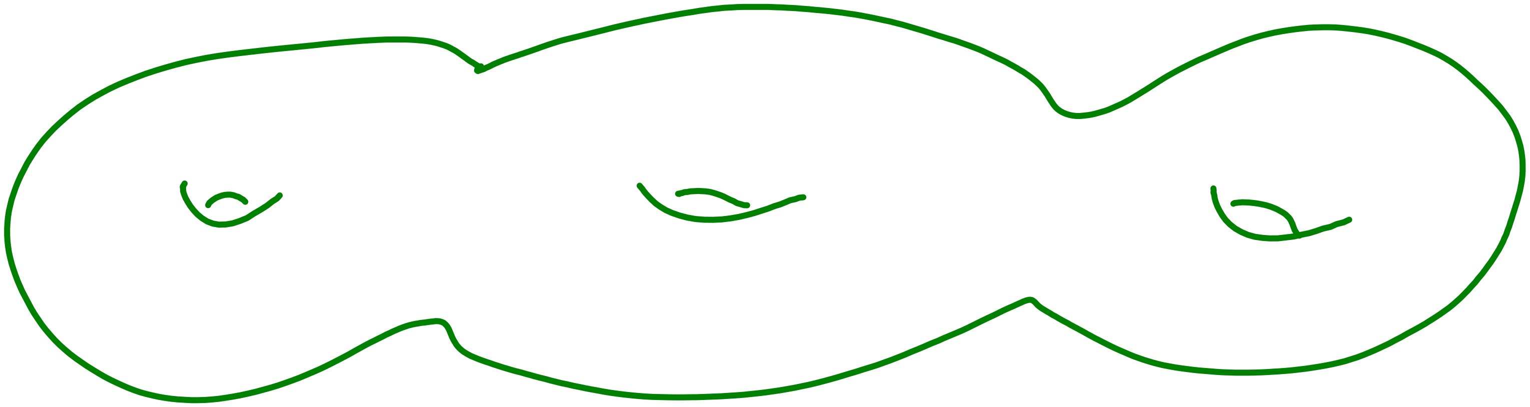


$$\pi_1(X, x) \cong \langle \gamma_1, \dots, \gamma_m \mid \gamma_1 \circ \cdots \circ \gamma_m = 1 \rangle$$

$$\cong \mathrm{Free}_{m-1} \quad \text{(Free group)}$$

m	0	1	2	3	4	5
π_1	1	1	$\mathbb{Z}$	Free_2	......	

E.g. X = genus g compact Riemann surface

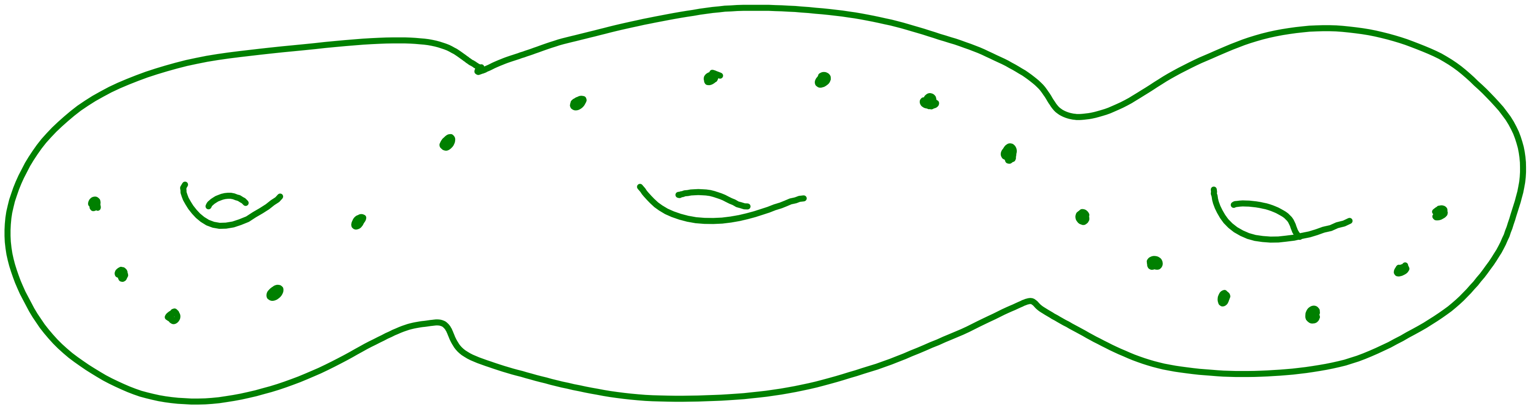


$$\pi_1(X) \cong \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1 \rangle$$

where $[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$

E.g. $\pi_1 \cong \mathbb{Z}^2$ if $g = 1$

E.g. X = m-punctured genus g compact Riemann surface



$$\pi_1(X) \cong \left\langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_m \;\middle|\; \prod_1^g [\alpha_i, \beta_i] \prod_1^m \gamma_j = 1 \right\rangle$$

"surface groups"

Nonabelian representations of surface groups arose in Riemann's work on the Gauss hypergeometric equation

B. Riemann 1857

Beiträge zur Theorie

der

durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$

darstellbaren Functionen

von

Bernhard Riemann,
Assessor der Königl. Gesellschaft der Wissenschaften.

Aus dem siebenten Bande der Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen.

Göttingen,
Verlag der Dieterichschen Buchhandlung.
1857.

Non abelian representations of surface groups arose in Riemann's work on the Gauss hypergeometric equation

$$z(1-z)\,y'' + (az+b)\,y' + cy = 0$$

[a, b, c constants, $y(z)$]

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- singular points $0, 1, \infty \in \mathbb{P}^1(\mathbb{C})$

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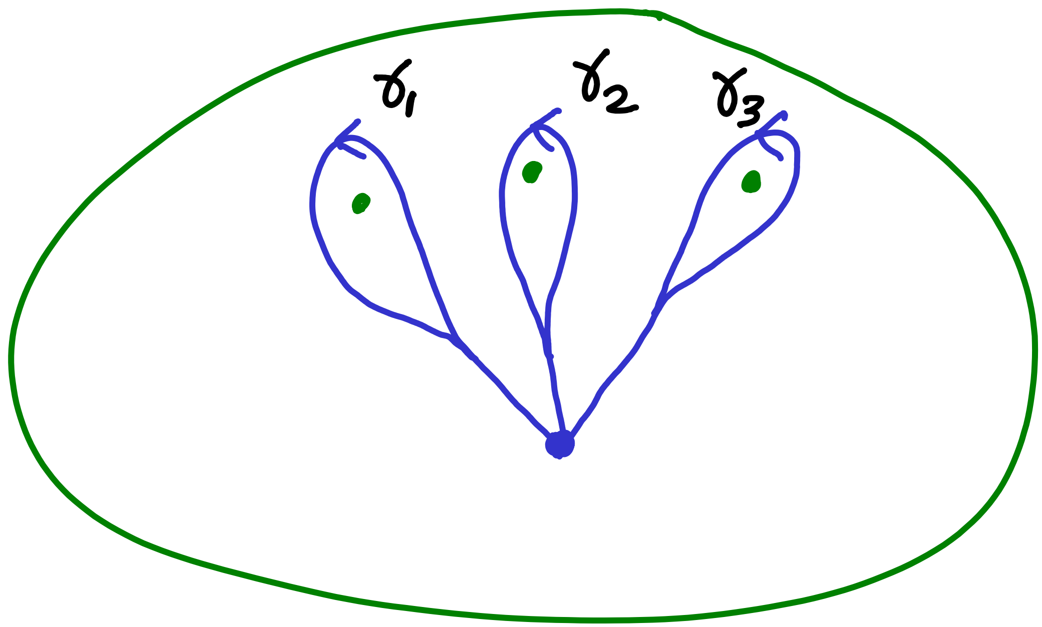
Riemann: Have basis of solutions on any disk $U \subset X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

- Look at "monodromy" of bases of solutions around loops

$$\Rightarrow \rho \in \mathrm{Hom}(\pi_1(X, x), GL_2(\mathbb{C}))$$

$$X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$$



$$M_i = \rho(\gamma_i)$$

$$\mathrm{Hom}(\pi_1(X), GL_2(\mathbb{C})) \cong \{ M_1, M_2, M_3 \in GL_2(\mathbb{C}) \mid M_1 M_2 M_3 = 1 \}$$

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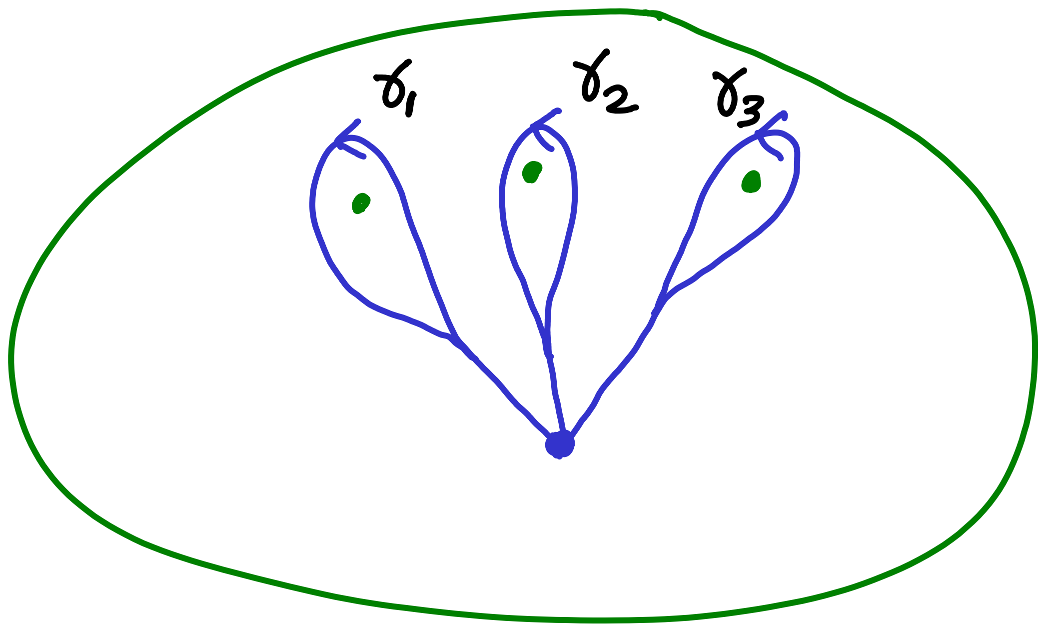

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- constants a, b, c $\sim$ conjugacy classes of M_1, M_2, M_3
- conjugacy class of ρ in $\mathrm{Hom}(\pi_1, G)/G$ is intrinsic

(indept. of basepoint and initial basis)

More generally taking monodromy gives map:

Order n linear differential equations with singular points $a_1, \ldots, a_m$

$$\Big\Downarrow \quad \text{"Riemann–Hilbert map"}$$

Point of $\quad \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}), GL_n(\mathbb{C})) / GL_n(\mathbb{C})$

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Hilbert's 21st problem (modern restatement):

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Hilbert's 21st problem (modern restatement):

What's going on here?

– is there a precise correspondence here somewhere?

Evolution 1

$$\text{Order } n \text{ differential equations} \underset{\text{cyclic vector}}{\overset{\text{basis}}{\rightleftarrows}} n \times n \text{ first order systems } \frac{d}{dz} - A$$

[A $n \times n$ matrix of mero. functions]

Evolution 2

$n \times n$ first order systems $\frac{d}{dz} - A$

connections on trivial rank n vector bundle $d - A\,dz$

(on $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$)

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$$\parallel$$
$$d - B$$

[B $n \times n$ matrix of mero. one-forms]

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$\|$

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[B $n \times n$ matrix of mero. one-forms]

Locally have fundamental solutions $\Phi : U \to GL_n(\mathbb{C})$

$$\boxed{d\Phi = B\,\Phi}$$

Example $a_1, \dots, a_m \in \mathbb{C}$ $A_1, \dots, A_m \in \mathrm{End}(\mathbb{C}^n)$

$$d - \sum_1^m \frac{A_i}{z - a_i}\, dz$$

$\sum A_i = 0$ (no pole at ∞)

Example $\quad a_1, \dots, a_m \in \mathbb{C} \qquad A_1, \dots, A_m \in \mathrm{End}(\mathbb{C}^n)$

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$$\Big\Downarrow \ \mathrm{RH}$$

$$\rho \in \mathrm{Hom}\left(\pi_1(\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}), GL_n(\mathbb{C})\right)$$

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Theorem (Bolibruch)

This Riemann-Hilbert map is <u>not</u> surjective in general

Evolution 3

connections on trivial rank n vector bundle (on $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$)

$$\nabla = d - B$$

connections ∇ on rank n vector bundles V (on $\Sigma \setminus \{a_1, \dots, a_m\}$)

Σ genus g Riemann surface

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$$\nabla : V \longrightarrow V \otimes \Omega^1$$

$$\nabla(fs) = (df)s + f(\nabla s)$$

Locally: $\nabla = d - B$

P. Deligne 1970

Lecture Notes in Mathematics

A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

163

Pierre Deligne
Institut des Hautes Etudes Scientifiques
Bures-sur-Yvette/France

Equations Différentielles à Points Singuliers Réguliers

Springer-Verlag
Berlin · Heidelberg · New York 1970

$\Sigma = \overline{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured Riemann surface

$G = GL_n(\mathbb{C})$

$\mathrm{Hom}(\pi_1(\Sigma), G) / G$

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$$\left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ holom. vector bundle} \\ \nabla \text{ holom. connection} \end{array} \right\} \Big/ \text{isom.} \quad \overset{\text{analytic Riemann-Hilbert}}{\underset{\cong}{\longleftrightarrow}} \quad \mathrm{Hom}(\pi_1(\Sigma), G) / G$$

$\Sigma = \overline{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured smooth algebraic curve$/\mathbb{C}$

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$$\left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \\ \text{with regular sing.s} \end{array} \right\} \Big/ \text{isom.} \xleftrightarrow[\text{Deligne Riemann-Hilbert}]{\simeq} \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

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restrict to Σ

$$\downarrow$$

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- $\boxed{\text{Representations of } \pi_1 \text{ classify algebraic differential equations (in this sense)}}$
- similarly for any smooth quasi-proj. var. (Deligne) < add "flat/integrable"; simple poles ⟶ logarithmic
- can now study transcendental aspects of RH map

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Aside: Applications/link to modern moduli theory

E.g. $m = 0$ (no poles), Σ compact smooth complex algebraic curve

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\{ \text{Alg. vector bundles } V \to \Sigma \} \big/ \text{isom.} &
\end{tikzcd}$$

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A. Weil: ① π is not onto

② $\pi \circ RH$ is injective on unitary representations

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\{ \text{Alg. vector bundles } V \to \Sigma \} / \text{isom.} & & \mathrm{Hom}(\pi_1(\Sigma), U_n) / U_n \arrow[ll] \arrow[u, bend right]
\end{tikzcd}$$

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"Weil's unitary trick"

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\{ \text{Alg. vector bundles } V \to \Sigma \}/\text{isom.} \arrow[d, phantom, "\cup"] & & \mathrm{Hom}(\pi_1(\Sigma), U_n)/U_n \arrow[ll] \arrow[u] \arrow[d, phantom, "\cup"] \\
\left\{ \begin{array}{l} \text{stable} \quad (\text{rk } n,\ \deg 0) \\ \text{alg. vector bundles } V \to \Sigma \end{array} \right\} \Big/ \text{isom.} \arrow[rr, leftrightarrow, "\cong"] & & \mathrm{Hom}^{irr}(\pi_1(\Sigma), U_n)/U_n
\end{tikzcd}$$

Mumford

Narasimhan – Seshadri

$V \to \Sigma$ is <u>stable</u> if

$$\frac{\deg(W)}{\mathrm{rank}(W)} < \frac{\deg(V)}{\mathrm{rank}(V)}$$

for any sub-bundle W of V

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(complex manifold + compatible symplectic structure)

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- so (via Naras-Sesh.) $\operatorname{Hom}^{irr}(\pi_1(\Sigma), U_n)/U_n$ is Kähler (complex manifold + compatible symplectic structure)
- similarly $\operatorname{Hom}^{irr}(\pi_1(\Sigma), GL_n(\mathbb{C}))/GL_n(\mathbb{C})$ is hyperKähler (Hitchin, Donaldson, Corlette, Simpson)

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Here only two complex structures are not isomorphic:

1 as complex algebraic connections or complex π_1 representations

2 as a moduli space of stable Higgs bundles $\sim T^*\{\text{stable vector bundles}\}$

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(Hitchin, Donaldson, Corlette, Simpson)

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1 as complex algebraic connections or complex π_1 representations

2 as a moduli space of stable Higgs bundles $\sim T^*\{$stable vector bundles$\}$

$$(E, \Phi) \quad \begin{cases} E \to \Sigma & \text{holom. vector bundle} \\ \Phi : E \to E \otimes \Omega^1 & (\mathcal{O}\text{-linear}) \end{cases}$$

$$\Phi(fs) = f\,\Phi(s) \quad (\text{degen. Leibniz})$$

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

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$$\Sigma \Longrightarrow H^1(\Sigma, G)$$

smooth compact curve

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$$\Sigma \Longrightarrow H^1(\Sigma, G) \text{ --- } \left\{ \text{Connections } (V, D) \right\} / \text{isom.}$$

smooth compact curve

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Fix $G = GL_n(\mathbb{C})$

$$
\begin{tikzcd}
\Sigma \arrow[r, Rightarrow] & H^1(\Sigma, G) \arrow[r, dash] \arrow[dr, dash] & \{\text{Connections } (V, \nabla)\}/\text{isom.} \arrow[d, equal, "\text{RH correspondence}"] \\
& & \mathrm{Hom}(\pi_1(\Sigma), G)/G
\end{tikzcd}
$$

smooth compact curve

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$$\underset{\substack{\text{smooth}\\ \text{compact}\\ \text{curve}}}{\Sigma} \Longrightarrow \begin{array}{c} \{\text{connections } (V, \nabla)\}/\text{isom.} \\ \Big\| \;\; \text{RH correspondence} \\ \mathrm{Hom}(\pi_1(\Sigma), G)/G \end{array}$$

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"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$$\Sigma \Longrightarrow \left\{ \text{stable connections } (V,\nabla) \right\} / \text{isom.}$$

smooth compact curve

$$\text{RH correspondence}$$

$$\mathrm{Hom}^{irr}(\pi_1(\Sigma), G)/G$$

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$\Sigma \Longrightarrow$

smooth compact curve

$$\mathcal{M}_{DR}(\Sigma) = \{\text{stable connections } (V, \nabla)\}/\text{isom.}$$

$$\Vert\wr \quad \text{RH isomorphism}$$

$$\mathcal{M}_B(\Sigma) = \mathrm{Hom}^{irr}(\pi_1(\Sigma), G)/G$$

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$\Sigma \Longrightarrow$

smooth compact curve

$$\mathcal{M}_{DR}(\Sigma)$$

$$\Big\| \wr \quad \text{RH isomorphism}$$

$$\mathcal{M}_B(\Sigma)$$

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$$\left\{ \begin{array}{c} \text{solutions to Hitchin's} \\ \text{equations} \end{array} \right\}$$

$$\Sigma \Longrightarrow \mathcal{M}(\Sigma) \underset{\text{Corlette-Donaldson}}{\cong} \mathcal{M}_{DR}(\Sigma)$$

smooth compact curve

Hyperkahler manifold

$$\mathcal{M}_{DR}(\Sigma) \cong \mathcal{M}_B(\Sigma) \quad \text{RH isomorphism}$$

so for $m=0$ get a rich picture:

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Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma) = \{\text{stable Higgs bundles } (E,\Phi)\}/\text{isom.}$$

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$$\mathcal{M}_{DR}(\Sigma) \;\big|\big|\big|\; \text{RH isomorphism} \;\; \mathcal{M}_B(\Sigma)$$

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Hitchin-Simpson $\qquad \mathcal{M}_{Dol}(\Sigma)$

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Hyperkahler manifold

Corlette-Donaldson

RH isomorphism

$\mathcal{M}_B(\Sigma)$

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Hitchin-Simpson

$\mathcal{M}_{Dol}(\Sigma)$

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Corlette-Donaldson

$\mathcal{M}_{DR}(\Sigma)$

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3 algebraic structures

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$$\begin{tikzcd}[column sep=large] & & \mathcal{M}_{Dol}(\Sigma) \arrow[d, equal, "\text{Non-abelian Hodge}"] \\ \Sigma \arrow[r, Rightarrow] & \mathcal{M}(\Sigma) \arrow[ur, equal, "\text{Hitchin-Simpson}"] \arrow[r, equal, "\text{Corlette-Donaldson}"'] & \mathcal{M}_{DR}(\Sigma) \arrow[d, equal, "\text{RH isomorphism}"] \\ & & \mathcal{M}_B(\Sigma) \end{tikzcd}$$

smooth compact curve

Hyperkahler manifold

3 algebraic structures

- Similarly for π_1 (punctured curve)

(Simpson, Konno, Nakajima ...)

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma)$$

Hitchin-Simpson $\equiv$

$$\Big\| \text{ Non-abelian Hodge}$$

$\Longrightarrow$ $\mathcal{M}(\Sigma)$ $\equiv$ $\mathcal{M}_{DR}(\Sigma)$

Hyperkähler manifold

Corlette-Donaldson

$$\Big\| \text{ RH isomorphism}$$

$$\mathcal{M}_B(\Sigma)$$

3 algebraic structures

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"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

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$$\mathcal{M}_{B}(\Sigma)$$

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma)$ — Algebraic integrable Hamiltonian systems (Hitchin)

$\wr\!|\!|$ Non-abelian Hodge

$\mathcal{M}_{DR}(\Sigma)$

$\wr\!|\!|$ RH isomorphism

$\mathcal{M}_{B}(\Sigma)$

$$\mathcal{M}_{Dol}(\Sigma) \downarrow \mathbb{H}$$

Fix $G = GL_n(\mathbb{C})$

"Non-abelian Hodge package"

$\mathcal{M}_{Dol}(\Sigma)$

$\cong$ Non-abelian Hodge

$\mathcal{M}_{DR}(\Sigma)$ — Isomonodromy systems (as Σ varies)

$\cong$ RH isomorphism

$\mathcal{M}_B(\Sigma)$

$$\begin{array}{c} \underset{\sim}{\Sigma} \\ \downarrow \\ \mathbb{B} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathcal{M}_{DR}(\Sigma_b) & \subset & \mathcal{M}_{DR/\mathbb{B}} \\ \downarrow & & \downarrow \\ b & \in & \mathbb{B} \end{array}$$

fibre bundle with flat nonlinear connection

e.g. Painlevé VI equations, Schlesinger system

"nonabelian Gauss-Manin connection"

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$$\begin{array}{l}\mathcal{M}_{Dol}(\Sigma) \\ \quad \Vert\wr \ \text{Non-abelian Hodge} \\ \mathcal{M}_{DR}(\Sigma) \\ \quad \Vert\wr \ \text{RH isomorphism} \\ \mathcal{M}_B(\Sigma)\end{array}$$

$\mathcal{M}_B(\Sigma)$ —— Nonlinear braid/mapping class group actions

$\pi_1(\mathcal{M}_{g,m})$

$$\begin{array}{c}\Sigma \\ \downarrow \\ \mathbb{B}\end{array} \Rightarrow \pi_1(\mathbb{B}, b) \circlearrowright \mathcal{M}_B(\Sigma_b)$$

by algebraic Poisson automorphisms

Wild nonabelian Hodge theory on curves

Wild nonabelian Hodge theory on curves

Choose

- $G = GL_n(\mathbb{C})$, $T \subset G$
- Σ compact smooth complex algebraic curve
- $a_1, \dots, a_m \in \Sigma$ distinct points

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Definition If $a \in \Sigma$, an irregular type Q at a is an element $Q \in \mathfrak{t}(\hat{K}) / \mathfrak{t}(\hat{\mathcal{O}})$

If z is a local coordinate vanishing at a

$$\hat{\mathcal{O}} = \mathbb{C}[\![z]\!] , \quad \hat{K} = \mathbb{C}((z))$$

$$Q = \frac{A_r}{z^r} + \cdots + \frac{A_1}{z} \quad \text{for some } A_i \in \mathfrak{t} = \mathrm{Lie}(T)$$

Wild nonabelian Hodge theory on curves

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"irregular curve" or "wild Riemann surface"
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- weights $\theta_1, \dots, \theta_m \in \mathfrak{t}_{\mathbb{R}} = X_*(T) \otimes \mathbb{R} \subset \mathfrak{t}$

  $\left((\theta_i)_{jj} \in [0,1) \quad j = 1, \dots, n \right)$

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Let $\mathfrak{h}_i = C_{\mathfrak{g}}(Q_i) \subset \mathfrak{g}$ (centraliser)

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Note that $\theta \in \mathfrak{t}_{\mathbb{R}}$ determines a parabolic $\mathfrak{p}_\theta \subset \mathfrak{g}$

$$\mathfrak{p}_\theta(\mathfrak{g}) = \left\{ X \in \mathfrak{g} \;\middle|\; \lim_{z \to 0} z^{\theta} X z^{-\theta} \text{ along any ray exists} \right\}$$

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$$\mathfrak{p}_\theta(\mathfrak{g}) = \mathrm{Stab}(\mathcal{F}_\theta), \qquad (\mathcal{F}_\theta)_\alpha = \bigoplus_{\beta \geq \alpha} E_\beta \quad \begin{pmatrix} \text{eigenspaces} \\ \text{of } \theta \end{pmatrix}$$

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& similarly $\mathfrak{p}_{\theta_i}(\mathfrak{h}_i) \subset \mathfrak{h}_i$ & $\mathfrak{l}_i$ is Levi of $\mathfrak{p}_{\theta_i}(\mathfrak{h}_i)$

Consider triples $(V, \nabla, \mathcal{F})$

- $V \longrightarrow \Sigma$ rank n holom. vector bundle
- $\nabla : V \longrightarrow V \otimes \Omega^1(*D)$ mero. connection $D = \sum a_i$
- $\mathcal{F} = (\mathcal{F}_i)_{i=1}^m$ flags in fibres $V_{a_1}, \ldots, V_{a_m}$

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such that:

Near a_i V has a local trivialization in which

- $\nabla = d - A$, $A = dQ_i + \Lambda_i \frac{dz}{z} + \text{holom.}$

for some $\Lambda_i \in \mathfrak{h}_i$

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- $\pi(\Lambda_i) \in \mathcal{O}_i \subset \mathfrak{l}_i$ $(\pi : \mathfrak{p}_{\theta_i}(\mathfrak{h}_i) \twoheadrightarrow \mathfrak{l}_i)$

Thm (Biquard–B. '04 building on Hitchin, Donaldson, Corlette, Simpson, Simpson, Nakajima, Sabbah, …)

The moduli space $\mathcal{M}_{DR}(\Sigma, \underset{\sim}{\theta}, \underset{\sim}{\alpha})$ of isomorphism classes of such mero. connections which are stable and parabolic degree zero is

- a hyperkähler manifold
- canonically diffeo. to a space of mero. Higgs bundles
- complete if $\underset{\sim}{\theta}, \underset{\sim}{\alpha}$ sufficiently generic

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- complete if $\underset{\sim}{\theta}, \underset{\sim}{\alpha}$ sufficiently generic

- Higgs fields should look like $-\frac{1}{2} dQ_i + \Lambda_i \frac{dz}{z}$ + holom. near a_i
- same 'rotation' of the weights / eigenvalues as in Simpson 1990

Simpson's table (JAMS '90) (notation & extension to other G / parahoric case, PB '10)

	Dolbeault/Higgs	DR/Connections	Betti/monod.
weights $t_{\mathbb{R}}$		θ	
eigenvalues $t_{\mathbb{C}}$		$\tau + \sigma$	

$t_{\mathbb{R}}$ $\quad$ $it_{\mathbb{R}}$

(eigenvalues of Λ)

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	Dolbeault/Higgs	DR/Connections	Betti/monod.	
weights $t_{\mathbb{R}}$		θ	$\phi = \theta + \tau$	
eigenvalues $t_{\mathbb{C}}$		$\tau + \sigma$	$\exp(2\pi i(\tau+\sigma))$	

τ: $t_{\mathbb{R}}$ (eigenvalues of Λ)

σ: $i t_{\mathbb{R}}$

$$\mathrm{Pardeg}(V,\nabla,\mathcal{F}) = \deg(V) + \sum_{1}^{m} \mathrm{Tr}\,\theta_i = \sum \mathrm{Tr}\,\Lambda_i + \mathrm{Tr}\,\theta_i = \sum \mathrm{Tr}\,\phi_i$$

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	Dolbeault/Higgs	DR/Connections	Betti/monod.	
weights $t_{\mathbb{R}}$	$-\tau$	θ	$\phi=\theta+\tau$	
eigenvalues $t_{\mathbb{C}}$	$-\frac{1}{2}(\phi+\sigma)$	$\tau+\sigma$	$\exp(2\pi i(\tau+\sigma))$	

(eigenvalues of Γ)

$t_{\mathbb{R}}$ $it_{\mathbb{R}}$

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	Dolbeault/Higgs	DR/Connections	Betti/monod.	
weights $t_{\mathbb{R}}$	$-\tau - [-\tau]$	θ	$\phi = \theta + \tau$	
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Sufficient stability conditions

If no strictly semistable points then $\mathcal{M}$ is complete

1. If (V', ∇') subconnection of (V, ∇)

$$\deg V' = \sum \operatorname{Tr} \Lambda_i' = \sum \operatorname{Tr}(\tau_i' + \sigma_i') \quad \in \mathbb{Z}$$

and $\operatorname{Tr} \tau_i' = \sum_{j \in S} (\tau_i)_{jj}$ for some $S \subset \{1, \ldots, \operatorname{rk} V\}$, $\# S = \operatorname{rk} V'$

$$\operatorname{Tr} \sigma_i' = \sum_S (\sigma_i)_{jj}$$

i.e. a "subsum" of $\sum_1^m \operatorname{Tr} \tau_i + \operatorname{Tr} \sigma_i$ is in $\mathbb{Z}$

(if $(\underset{\sim}{\tau}, \underset{\sim}{\sigma})$ off of these hyperplanes then $\mathcal{M}$ complete)

Fix $G = GL_n(\mathbb{C})$

Σ

irregular curve

("wild Riemann surface")

Fix $G = GL_n(\mathbb{C})$

smooth compact curve

$(a_1, \ldots, a_m)$ m distinct points of Σ

$$\boldsymbol{\Sigma} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$$

$(Q_1, \ldots, Q_m)$ Q_i irregular type at a_i

irregular curve

("wild Riemann Surface")

Fix $G = GL_n(\mathbb{C})$

$$\boldsymbol{\Sigma} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Longrightarrow \begin{array}{c} \mathcal{M}_{DR}(\boldsymbol{\Sigma}) \\ \Big\| \wr \; \text{RH irregular isomorphism} \\ \mathcal{M}_B(\boldsymbol{\Sigma}) \end{array}$$

irregular curve

Fix $G = GL_n(\mathbb{C})$

$$\boldsymbol{\Sigma} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Longrightarrow \begin{array}{c} \mathcal{M}_{DR}(\boldsymbol{\Sigma}) \\ \wr\!\wr\!\wr \quad \text{RH irregular isomorphism} \\ \mathcal{M}_B(\boldsymbol{\Sigma}) \end{array}$$

irregular curve

Locally near a_i:

$$\nabla \cong d - (\underbrace{dQ_i}_{\text{irregular part specified by irregular type}} + \Lambda \frac{dz}{z} + \cdots)$$

WLOG $\Lambda \in H_i := C_g(Q_i)$

Fix $G = GL_n(\mathbb{C})$

conjugacy class (& weights)

$\mathcal{C} \subset \underset{\sim}{H} = H_1 \times \cdots H_m \quad \left(H_i = C_G(Q_i)\right)$

$\Sigma = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Longrightarrow$

irregular curve

$$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$$
$$\Big|\Big|\Big|\wr \quad \text{irregular RH isomorphism}$$
$$\mathcal{M}_B(\Sigma, \mathcal{C})$$

Locally near a_i:

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Fix $G = GL_n(\mathbb{C})$

$\bullet\ \bullet\ \bullet\ \bullet$

Fix $G = GL_n(\mathbb{C})$

(weighted) conjugacy class

$\mathcal{C} \subset \underset{\sim}{H}$

$\Sigma \Longrightarrow \mathcal{M}(\Sigma, \mathcal{C})$

irregular curve

Hyperkahler manifold

"Wild Hitchin space"

(Biquard-B. '04)

Hitchin-Simpson

Biquard-B.

Corlette-Donaldson

Sabbah

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$\wr\!\wr\!\wr$ Wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\wr\!\wr\!\wr$ irregular RH isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

(See e.g. survey 1203.6607 for full details)

Fix $G = GL_n(\mathbb{C})$

njugacy class

c $\underset{\sim}{H}$

lar

e

$\Longrightarrow$ $\mathcal{M}(\Sigma, \mathcal{C})$ $\cong$ $\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

Hitchin-Simpson

Biquard-B.

Hyperkahler manifold

"wild Hitchin space"

(Biquard-B. '04)

$\mathcal{M}(\Sigma, \mathcal{C}) \cong \mathcal{M}_{DR}(\Sigma, \mathcal{C})$

Corlette-Donaldson Sabbah

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$\cong$ Wild non-abelian Hodge isom.

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$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

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$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$\Big|\Big|\Big|\wr$ Wild non-abelian Hodge isom.

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$\Big|\Big|\Big|\wr$ RH irregular isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

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4)

ull details)

Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$$

$\Big\|\!\Big\wr$ wild non-abelian Hodge isom.

$$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$$

$\Big\|\!\Big\wr$ RH irregular isomorphism

$$\mathcal{M}_{B}(\Sigma, \mathcal{C})$$

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$ — Algebraic integrable systems (Hitchin, Nitsure, Bottacin, Markman...)

$\parallel\!\!\wr$ wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\parallel\!\!\wr$ RH irregular isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

$$\mathcal{M}_{Dol}(\Sigma, \mathcal{C}) \hookrightarrow \mathcal{M}_{Dol}(\Sigma, \mathcal{C})$$

$$\downarrow$$

$$\mathbb{H}$$

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$|||$ wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$ — Isomonodromy systems (as Σ varies in admissible fashion)

$|||$ irregular RH isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

$$\begin{array}{ccccc} \underset{\sim}{\Sigma} & & \mathcal{M}_{DR}(\Sigma_b) & \subset & \mathcal{M}_{DR/\mathbb{B}} \\ \downarrow & \Rightarrow & \downarrow & & \downarrow \\ \mathbb{B} & & b & \in & \mathbb{B} \end{array}$$

— fibre bundle with flat nonlinear connection

e.g. Painlevé equations, Schlesinger system,

JMU system, Simply-laced isomonodromy systems

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$\|\!\!\wr$ wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\|\!\!\wr$ irregular RH isomorphism

$\mathcal{M}_B(\Sigma, \mathcal{C})$ —— Nonlinear braid/mapping class group actions

"Wild mapping class groups"

e.g. Braiding of Stokes data of Cecotti-Vafa/Dubrovin

$$\begin{array}{c}\Sigma \\ \downarrow \\ \mathbb{B}\end{array} \Rightarrow \pi_1(\mathbb{B}, b) \circlearrowright \mathcal{M}_B(\Sigma_b)$$

by algebraic Poisson automorphisms

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607)

					$2+2$	$2+2$	$2+2$			
					2	2	2			
E_8	E_7	E_6	D_4	$A_3 = D_3$	D_2	D_1	D_0	A_2	A_1	A_0
6	4	3	2	2				2	2	2
$1+1+1$	$1+1+1$	$1+1+1$	$1+1+1+1$	$2+1+1$				$3+1$	4	4

affine Weyl group

minimal rank of bundles

pole orders

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607)

Tame ⟵ ¦ ⟶ Wild

	Tame				Wild						
affine Weyl group	E_8	E_7	E_6	D_4	$A_3 = D_3$	D_2	D_1	D_0	A_2	A_1	A_0
minimal rank of bundles	6	4	3	2	2	2	2	2	2	2	2
pole orders	1+1+1	1+1+1	1+1+1	1+1+1+1	2+1+1	2+2	2+2	2+2	3+1	4	4

affine Weyl group

minimal rank of bundles

pole orders

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607)

E_8 E_7 E_6

P_3 P_3' P_3''

D_2 D_1 D_0

D_4 $A_3 = D_3$

P_6 P_5

A_2 A_1 A_0

P_4 P_2 P_1

Phase spaces for Painlevé differential equations

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607)

$\mathcal{M}^* \cong$ ALE

$\mathcal{M}^* \cong$ ALF

E_8 E_7 E_6

D_4 $A_3 = D_3$

D_2 D_1 D_0 Atiyah-Hitchin

A_2 A_1 A_0 $\mathbb{C}^2$

[$\mathcal{M}^* \subset \mathcal{M}$ open piece where bundle holom. trivial]