Construction of Haar measure (contd)

We had shown that μ^* is a left-invariant measure on $\mathscr{A}$. We will now show that μ^* gives a finite measure to every compact set and a positive measure to every nonempty open set.

Suppose $K \subseteq G$ is compact. $\mathcal{U}_0$ covers K, so some finitely many sets in $\mathcal{U}_0$, say $U_1, \ldots, U_l$ cover K. Since λ of every set in $\mathcal{U}_0$ is finite,

$$\mu^*(K) \leq \sum_{i=1}^{l} \lambda(U_i) < \infty$$

Suppose $U \subseteq G$ is open, and $U \neq \emptyset$. Using local compactness, get nonempty open H such that $H \subseteq \overline{H} \subseteq U$ and $\overline{H}$ is compact. For calculating $\mu^*(\overline{H})$, it suffices to consider only finite covers, since any countably infinite cover has a finite subcover, which gives only a better (lower) potential value for $\mu^*(\overline{H})$. Let $\epsilon > 0$ be given. Get $U_1, \ldots, U_l \in \mathcal{U}_0$ covering $\overline{H}$ such that $\sum_{i=1}^{l} \lambda(U_i) < \mu^*(\overline{H}) + \epsilon$. Note that λ is defined only on $\mathcal{U}$.

$$\begin{aligned}
\lambda(H) &\leq \lambda\left(\bigcup_{i=1}^{l} U_i\right) && \text{(monotonicity)} \\
&\leq \sum_{i=1}^{l} \lambda(U_i) && \text{(finite subadditivity)} \\
&\leq \mu^*(\overline{H}) + \epsilon \\
&\leq \mu^*(U) + \epsilon && \text{(monotonicity)}
\end{aligned}$$

Since this holds for all $\epsilon > 0$, $\mu^*(U) \geq \lambda(H) > 0$.

μ^* is a left-invariant Haar measure on the Borel σ-field of G.

Uniqueness

In our current setup (a locally compact metrizable topological group G), we will show that if μ and ν are two left-invariant Haar measures, then μ can be scaled so that for all continuous functions with compact support $f : G \to \mathbb{C}$, we have $\int f \, d\mu = \int f \, d\nu$. A positive finite scalar multiple of a left-invariant Haar measure is also a left-invariant Haar measure.

The proof we give is a tricky proof due to Von Neumann, though more straightforward proofs exist.

Suppose μ, ν are left-invariant Haar measures on G. Fix $\phi : [0, \infty)$, a continuous function with compact support which is not the identically zero function. This can be done using the metric and local compactness. ϕ is necessarily bounded. If ϕ takes the value a somewhere, with $a > 0$, then $\{x : \phi(x) > a/2\}$ is nonempty open, so has positive measure. $\int \phi \, d\mu > 0$, and $\int \phi \, d\nu > 0$. We may scale μ and ν such that

$$\int \phi \, d\mu = \int \phi \, d\nu = 1$$

Define $\Delta : G \to [0, \infty)$,

$$\Delta(g) = \int \phi(xg^{-1}) \, d\mu(x)$$

Lemma. Δ is well-defined and is a continuous function of compact support.

Proof. To show that Δ is well-defined, we need to show that for each $g \in G$, the integral used to define $\Delta(g)$ is finite. Let $K \subseteq G$ be a compact set such that ϕ is zero outside K. Kg is also compact, and so has finite μ measure (but $\mu(Kg)$ need not be equal to $\mu(K)$; μ is only left-invariant). To obtain $\Delta(g)$ we are integrating a bounded nonnegative function which is zero outisde Kg, and so $\Delta(g)$ is finite.

To show continuity of Δ, let $\{g_n\}_{n=1}^{\infty}$ be a sequence converging to g. We will show that

$$\tilde{K} = Kg \cup \bigcup_{n \in \mathbb{N}} Kg_n$$

is compact. Take any sequence in $\tilde{K}$. We need to find a convergent subsequence. If infinitely many terms occur in Kg or in some Kg_n, then we are done. Otherwise, only finitely many terms are in Kg, and only finitely many terms are in Kg_n, for each n. Then we can find a subsequence whose kth term is in Kg_{n_k}, where $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence. From this we can extract a further subsequence which converges to a point in Kg.

Now we can show that $\{\Delta(g_n)\}_{n=1}^{\infty}$ converges to $\Delta(g)$ using the usual DCT argument, using $M1_{\tilde{K}}$ as the integrable bound, where M is a bound on ϕ. $\square$

Lemma. For all $f : G \to \mathbb{C}$ continuous functions with compact support,

1. $\int f(x)\ d\mu(x) = \int f(x^{-1})\Delta(x^{-1})\ d\mu(x)$.
2. $\int f(x)\ d\mu(x) = \int f(x^{-1})\Delta(x^{-1})\ d\nu(x)$.
3. $\int f(x)\ d\mu(x) = \int f(x)\Delta(x)\Delta(x^{-1})\ d\mu(x)$.
4. $\int f(x)\ d\mu(x) = \int f(x)\Delta(x)\Delta(x^{-1})\ d\nu(x)$.
5. $\int f(x)\ d\mu(x) = \int f(x)\ d\nu(x)$.

Proof. We will prove 1 and 2 simultaneously. Fubini's theorem will be used freely. Everything is bounded and has compact support, so is integrable. In what follows, ρ can be taken to be either μ or ν. $\rho = \mu$ proves 1 and $\rho = \nu$ proves 2.

$$\begin{aligned}
&\int f(x)\ d\mu(x) \\
&= \left[\int \phi(y)\ d\rho(y)\right]\left[\int f(x)\ d\mu(x)\right] \\
&= \int \left[\int f(x)\ d\mu(x)\right]\phi(y)\ d\rho(y) \\
&= \int \left[\int f(y^{-1}x)\ d\mu(x)\right]\phi(y)\ d\rho(y) && \text{(replace } x \text{ by } y^{-1}x\text{, use left-invariance of } \mu\text{)} \\
&= \int \left[\int f(y^{-1}x)\phi(y)\ d\rho(y)\right]\ d\mu(x) \\
&= \int \left[\int f(y^{-1})\phi(xy)\ d\rho(y)\right]\ d\mu(x) && \text{(replace } y \text{ by } xy\text{, use left-invariance of } \rho\text{)} \\
&= \int f(y^{-1})\left[\int \phi(xy)\ d\mu(x)\right]\ d\rho(y) \\
&= \int f(y^{-1})\Delta(y^{-1})\ d\rho(y)
\end{aligned}$$

We now prove 3 and 4, using 1 and 2. Again, ρ can be taken to be either of μ or ν. $\rho = \mu$ proves 3 and $\rho = \nu$ proves 4.

$$\int f(x)\Delta(x)\Delta(x^{-1})\ d\rho(x)$$

$$
\begin{aligned}
&= \int g(x^{-1})\Delta(x^{-1})\ d\rho(x) && \text{(where } g(x) = f(x^{-1})\Delta(x^{-1}))^1 \\
&= \int g(x)\ d\rho(x) && \text{(using 1 or 2)} \\
&= \int f(x^{-1})\Delta(x^{-1})\ d\rho(x) \\
&= \int f(x)\ d\rho(x) && \text{(using 1 or 2)}
\end{aligned}
$$

3 gives $\Delta(x)\Delta(x^{-1}) = 1$ for all $x \in G$. If not, suppose for some x, $\Delta(x)\Delta(x^{-1}) = a$ with $a > 1$ ($a < 1$ is similar). Let $U = \left\{x : \Delta(x)\Delta(x^{-1}) > \frac{1+a}{2}\right\}$. U is nonempty open. By choosing a suitable continuous function f with compact support contained in U, we can get a contradiction to 3.

Since $\Delta(x)\Delta(x^{-1}) = 1$ for all $x \in G$, 4 gives 5. $\square$

Suppose we know that μ and ν are regular from below, that is,

$$\rho(A) = \sup\{\rho(K) : K \text{ compact}, K \subseteq A\}$$

for $\rho \in \{\mu, \nu\}$. For example, this is true if (G, d) is complete and separable. If K is a compact set, we can get a sequence of continuous functions of compact support decreasing to 1_K, each bounded between 0 and 1. DCT will give $\mu(K) = \nu(K)$. Then regularity will give $\mu = \nu$.