* Purpose of the paper is to extend a modified version of a result of Vande Vate (1989) which characterizes stable matchings as the extreme points of some polytope.
Vande Vate gave an LP formulation for the standard stable matching problem b/w n men & n women each of who has a strictly ordered preference list which doesn't leave anybody of the opposite gender out.
In this paper, the author also considers stable matchings that allow staying single (unmatched), so $|M| = |W|$ is not necessary.

M : men , W : women

For each m, $W_m \subseteq W$ is the (strict) preference list of m
For each w, M_w is similarly defined.

For $w_1, w_2 \in W_m$, $w_1 >_m w_2$ means m prefers w_1 to w_2
$>_w$ is similarly defined on M_w

$<_k$, $\leq_k$, $\geqslant_k$ have the usual meaning for $k \in M \cup W$
'$<_k$ or $=$'

W_m is <u>acceptable</u> to m
M_w " " " w

$A = \{(m, w) \mid m \in M_w \text{ \& } w \in W_m\} \leftarrow$ <u>acceptable pairs</u>

A matching from M to W is represented by an assignment matrix $x = \{x_{ij}\}_{i \in M, j \in W}$ whose elements are all integral & we want to stipulate that

$$x_{ij} = \begin{cases} 1 & \text{if } i \text{ \& } j \text{ are matched to each other} \\ 0 & \text{otherwise} \end{cases}$$

- This is what we want the matrix x is characterize.
So, we define the following constraints :

$$\sum_{j \in W} x_{ij} \leq 1 \text{ for all } i \in M \quad \text{(1)}$$

{Each row in x can have at most one '1' value which is to ensure each man is matched to at most one woman}

Similarly to ensure each woman can be matched to at most one man in a matching, we have

$$\sum_{i \in M} x_{ij} \leq 1 \text{ for all } j \in W \quad \text{— (2)}$$

<u>Note</u>: We are considering at most one mate because one is allowed to remain unmatched here.

$$x_{ij} \geq 0 \quad \forall\, i \in M \ \& \ j \in W \quad \text{— (3)}$$ {Don't allow negative values}

So, if x is integer (meaning each value is integral) then clearly, $x_{ij} \in \{0,1\}$.

$w(x,m)$: mate of m in x
$m(x,w)$: mate of w in x

* A matching is <u>stable</u> if no one is matched to an unacceptable person & the following holds:

Say x is the assignment matrix representing the matching

There is no $(m,w) \in A$ s.t. (the pair is acceptable)

(i) both m & w have mates in x but $w >_m w(x,m)$ and $m >_w m(x,w)$ // Both prefer each other to their mates

(ii) m has a mate, w doesn't but $w >_m w(x,m)$
// m prefers w to his current mate

(iii) w has a mate, m doesn't but $m >_w m(x,m)$
// w prefers m to her current mate

(iv) both m & w don't have mates.

These conditions for stability can be expressed by the following constraints:

$$x_{ij} = 0 \quad \forall\, (i,j) \in (M \times W) \setminus A \quad \text{— (4)}$$ {Unacceptable pairs can't be matched}

Conditions (i), (ii), (iii), (iv) can be incorporated by the following constraint:

$$\sum_{\substack{j \in W_m \\ j >_m w}} x_{mj} + \sum_{\substack{i \in M_w \\ i >_w m}} x_{iw} + x_{mw} \geq 1 \quad \text{— (5)}$$

This inequality ensures that for a given stable matching at least one of the 3 terms must be 1, i.e., either

$x_{mw} = 1$ // m is matched to w OR

$\sum_{j >_m w} x_{mj} = 1$ // m prefers his mate to w

$\sum_{i >_w m} x_{iw} = 1$ // w prefers her mate to m

allowing partial or fractional matching

* Lemma 1 : let x be a matching. Then x is stable if & only if x satisfies (4) & (5).

Pf: It is evident that (5) is violated iff for some $(m,w) \in A$

$$\sum_{j >_m w} x_{mj} = \sum_{i >_w m} x_{iw} = x_{mw} = 0$$

which precisely characterizes one or more of the 4 conditions for violating stability.

* By lemma 1, it follows that $x = \{x_{ij}\}_{i \in M, j \in W}$ represents a stable matching iff x is an integer sol^n of (1) – (5)

Now we will see some results about the solutions (not necessary integral) of (1) – (5)

let x satisfy (1) – (5). Definitions :

$S_M(x) = \{m \in M \mid \sum_{j \in W} x_{mj} > 0\}$ // men who are matched (maybe partially) in x

$S_W(x) = \{w \in W \mid \sum_{i \in M} x_{iw} > 0\}$ // women who are matched in x (maybe partially or fractionally)

$w^*(x,m),\ w_*(x,m) \in \{j \in W \mid x_{mj} > 0\}$

Out of all partners of m in x, $w^*(x,m)$ is the best partner, i.e., most preferred w.r.t. $>_m$ order for $m \in S_M(x)$

$m^*(x,w)$ & $m_*(x,w)$ are similarly defined for $w \in S_W(x)$.

* Note: (4) assures that $\{j \in W \mid x_{mj} > 0\} \subseteq W_m$ so $>_m$ is defined on it.

* Observe that x is integer iff $\forall\, m \in S_m(x)$,

$$\sum_{j \in W} x_{mj} = 1 \;\&\; w^*(x,m) = w_*(x,m) \quad \text{i.e.,}$$

For any man who is matched, he is completely matched & has exactly one mate $w(x,m) = w^*(x,m) = w_*(x,m)$. We can switch the roles of m & w & say the same thing.

Lemma 2: Let x satisfy (1) – (5) & $(m,w) \in A$. Then

$m \notin S_m(x)$ or $m \in S_m(x)$ & $w \geq_m w^*(x,m)$

$$\Rightarrow \sum_{i \in M} x_{iw} = 1 \;\&\; m \leq_w m_*(x,w) \qquad \text{—— (6)}$$

Consider $m \notin S_m(x)$, so m is not matched at all, then clearly w must be fully matched otherwise it will violate (5), intuitively, we can add more fractional values to the mates of w in the matching to make it fully matched. So, $\sum_{i \in M} x_{iw} = 1$

Also, since m is not matched at all, $m \neq m_*(x,w)$ & $m >_w m_*(x,w)$ is also not possible as it violates stability $\because$ w prefers m to her worst partner but isn't matched to m. In case of an integral stable matching, it is easy to see, in case of a fractional matching $m >_w m_*(x,w)$ will violate (5).

So $m <_w m_*(x,w)$

Now suppose $m \in S_m(x)$ & $w \geq_m w^*(x,m)$, i.e., m is matched & w is at least as preferred as the best partner of m in x. So w must be fully matched as otherwise it would violate stability or more formally, (5) for similar reasons as previously seen.

Also, $m \leq_w m_*(x,w)$ because w is equally or more preferred to the best mate of m by m, so to avoid unstability, m must be at most as liked as the worst partner of w & not more than that. Hence ⑥ holds.

$$m \in S_m(x) \ \& \ w = w^*(x,m) \iff \sum_{i \in M} x_{iw} = 1 \ \& \ m = m_*(x,w) \quad \text{—— } (7)$$

Pf: ($\Rightarrow$) Suppose m is matched & w is the best mate of m. Then by the previous result ⑥, it follows that $\sum_{i \in M} x_{iw} = 1$, i.e., w is fully matched &

$m \leq_w m_*(x,w)$, i.e., w prefers m not more than her worst mate.

Suppose $m <_w m_*(x,w)$, but $\because$ w is the best mate of m, m is matched to w, i.e., $x_{mw} > 0$ which by definition means that $m \geq_w m_*(x,w) \Rightarrow \Leftarrow$

Hence, $m = m_*(x,w)$ //

* Now we derail a bit & using this result, we prove that m is matched in x iff m is fully matched in x

<u>Subclaim</u>:

$$m \in S_M(x) \iff \sum_{j \in W} x_{mj} = 1 \quad \text{—— } (f_m)$$

Pf of <u>subclaim</u>: ($\Leftarrow$) direction is trivial.

($\Rightarrow$): Define a set $F_W(x) = \{ w \in W \mid \sum_{i \in M} x_{iw} = 1 \}$ // fully matched women

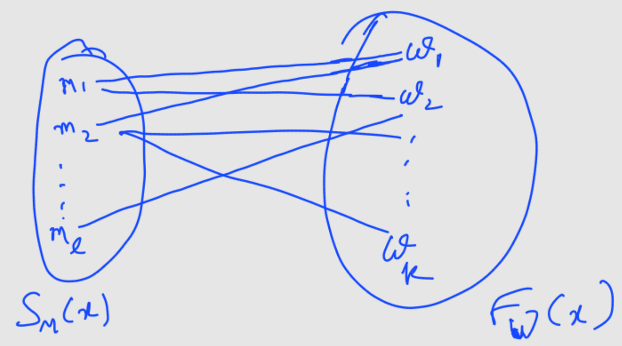
want to prove: $w^*(x,\cdot) : S_M(x) \to F_W(x)$ is a bijection.

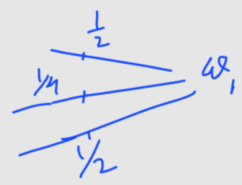
By the previously proven result, it is clear that $\forall m \in S_M(x)$, $w^*(x,m) \in F_W(x)$. Also observe $w^*(x,\cdot)$ is injective because suppose $m_1, m_2 \in S_M(x)$ s.t. $w^*(x,m_1) = w^*(x,m_2) = w$

By previous result, we know $m_1 = m_*(x,w)$ & $m_2 = m_*(x,w)$

$\Rightarrow m_1 = m_2 = m_*(x,w)$ $\because$ the worst partner of w is unique. Hence, $|F_W(x)| \geq |S_M(x)|$

Consider the corresponding bipartite graph of the matching x (not necessarily integral):





We know that for each $w \in F_W(x)$, $\sum_{i \in M} x_{iw} = 1$

Hence in this graph restricted to $S_M(x)$ & $F_W(x)$, the sum of all values of edges $= |F_W(x)|$. This gives 2 conclusions: i) $|F_W(x)| = |S_M(x)|$ because, since the edges are incident on elements in $S_M(x)$, we can take the sum of all edges from the left side & it should give $k = |F_W(x)|$ but if $|S_M(x)| < k$, then we know that edges outgoing from a single $m \in S_M(x)$ can add up to at most 1, sum over all $m \in S_M(x)$ can add up to at most $|S_M(x)| < |F_W(x)| = k \Rightarrow \in$.

ii) $\sum_{j \in W} x_{mj} = 1$, i.e., each $m \in S_M(x)$ must be fully matched since their edges must sum up to $k = |S_M(x)| = |F_W(x)|$ & edges for each man can sum up to at most 1. This establishes (fm)

$\because$ $w^*(x, \cdot)$ is injective & $|S_M(x)| = |F_W(x)|$,
$w^*(x, \cdot)$ is bijective.

Now, we are ready to prove ($\Leftarrow$) of (7).

Suppose w is fully matched, i.e., $\sum_{i \in M} x_{iw} = 1$ & $m = m_*(x, w)$

Then $w \in F_W(x)$ but $\because$ $w^*(x, \cdot)$ is onto, $\exists\, m' \in S_M(x)$ s.t. $w = w^*(x, m')$, i.e., w is the best partner of some $m' \in S_M(x)$

Then, we can apply ($\Rightarrow$) of (7) to say $m' = m_*(x, w) = m$

Hence, $m' = m \in S_M(x)$ & $m = m_*(x, w)$ holds. $\square$.

So, (7) holds.

Consider $\displaystyle\sum_{j >_m w} x_{mj} + \underbrace{\sum_{i >_w m} x_{iw} + x_{mw}} = 1$ —— (8)

If (8) holds, then converse of (6) holds as well.

It is easy to see that if w is fully matched & $m \le_w m_*(x, w)$, i.e., m is at most as liked as the worst mate of w in x, then $\displaystyle\sum_{i \geq_w m} x_{iw} = 1$

$\Rightarrow \displaystyle\sum_{j >_m w} x_{mj} = 0$ which clearly implies that either m is unmatched or if m is matched, his best mate is at most as good as w, i.e. $w^*(x, m) \le_m w$. ☐

Lemma 3 is just lemma 2 with roles of m & w reversed, it is true by symmetric args.

<u>Lemma 3</u>: let x satisfy (1) - (5) & $(m, w) \in A$. Then

$w \notin S_W(x)$ or $w \in S_W(x)$ & $m \geq_w m^*(x, w)$

$$\Rightarrow \sum_{j \in W} x_{mj} = 1 \ \& \ w \le_m w_*(x, m) \quad \text{—— (9)}$$

&

$$w \in S_W(x) \ \& \ m = m^*(x, w) \iff \sum_{j \in W} x_{mj} = 1 \ \& \ w = w_*(x, m) \quad \text{—— (10)}$$

$w \in S_W(x)$ iff $\displaystyle\sum_{i \in M} x_{iw} = 1$ —— (f_w)

If (8) holds then converse of (9) holds.

Now we are ready to prove the main result.

<u>Theorem 1</u>: let C be the solutions of (1) - (5). Then the integer points in C are precisely its extreme points.

Pf: ($\Rightarrow$) Easy, Suppose x is an integer sol^n of (1) - (5). Then x is an integer solution of (1) - (3) which are constraints for the general bipartite matching & for that we have already seen in class that integer points satisfying (1) - (3) are precisely the extreme points of the corresponding polytope. (Also can be derived by Birkhoff's theorem)

Hence, x is an extreme point of ① – ③ & ∴ an extreme point of ① – ⑤. //

(⟸): The tricky part. Suppose x is an extreme point of C. We define 3 matrices of order $|M| \times |W|$.

For $(m, w) \in (M, W)$,

$$(z^*)_{mw} := \begin{cases} 1, & \text{if } m \text{ is matched in } x \text{ \& } w \text{ is his best mate} \\ & [m \in S_m(x) \text{ \& } w = w^*(x,m)] \\ 0, & \text{otherwise} \end{cases} \qquad \text{—— (11)}$$

$$(z_*)_{mw} := \begin{cases} 1, & \text{if } m \text{ is matched in } x \text{ \& } w \text{ is his worst mate} \\ & [m \in S_m(x) \text{ \& } w = w_*(x,m)] \\ 0, & \text{otherwise} \end{cases} \qquad \text{—— (12)}$$

$$(z)_{mw} = (z^*)_{mw} - (z_*)_{mw} \qquad \text{—— (13)}$$

If we show that $z = 0$, then that implies $z^* = z_*$ which means, $\forall m \in S_m(x)$, i.e., for all men who are matched, their best & worst partners are the same i.e., $w^*(x,m) = w_*(x,m)$ hence m has exactly one partner & by (f_m), we know m is fully matched so by $(*)$, x is integer. So, it is enough to prove $z = 0$.

Also note, by using lemma 2 & 3, we can equivalently define z^* & z_* as:

$$(z^*)_{mw} = \begin{cases} 1 & \text{if } w \in S_W(x) \text{ \& } m = m_*(x,w) \\ 0, & \text{o/w} \end{cases} \qquad \text{—— (14)}$$

$$(z_*)_{mw} = \begin{cases} 1 & \text{if } w \in S_W(x) \text{ \& } m = m^*(x,w) \\ 0, & \text{o/w} \end{cases} \qquad \text{—— (15)}$$

Observe how z^* & z_* are formed:

In x, For all the men who are matched (maybe fractionally)

z^* matches them to only their best partners, fully. Similarly, z_* matches them to only their worst partners fully. Similarly from the side of matched women but "best" & "worst" gets interchanged.

We prove the following:

$$\sum_{j\in W} x_{ij}=1 \Rightarrow \sum_{j\in W} z_{ij}=0, \quad i\in M \qquad \text{—— (16)}$$

$$\sum_{i\in M} x_{ij}=1 \Rightarrow \sum_{i\in M} z_{ij}=0, \quad j\in W \qquad \text{—— (17)}$$

$$x_{ij}=0 \Rightarrow z_{ij}=0, \quad i\in M, j\in W \qquad \text{—— (18)}$$

$$z_{ij}=0, \quad (i,j)\in (M\times W)\setminus A \qquad \text{—— (19)}$$

For $(m,w)\in A$

$$\underbrace{\sum_{j>_m w} x_{mj}+\sum_{i>_w m} x_{iw}+x_{mw}=1}_{(21)} \Rightarrow \overbrace{\sum_{j>_m w} z_{mj}+\sum_{i>_w m} z_{iw}+z_{mw}=0}^{(22)} \qquad \text{—— (20)}$$

If (16) - (20) hold, then for sufficiently small ε, $x-\varepsilon z$ & $x+\varepsilon z$ satisfy (1) - (5) & hence $\in C$

Then $x=\frac{1}{2}(x+\varepsilon z)+\frac{1}{2}(x-\varepsilon z)$ but $\because$ x is extreme, $z=0$ & we are done!

It remains to prove (16) - (20)

(16) is easy to see because if some m is fully matched in x, then it is fully matched to one partner in z^* & in z_* & the difference of their sums would be zero as 1 & -1 will cancel each other out.

(17) can be argued similarly but from the POV of women.

In (18), if $x_{ij}=0$ then either i is completely unmatched in which case i is completely unmatched in z^* & z_* also & if i is matched, then j can't be his best or worst partner so $(z^*)_{ij}=0=(z_*)_{ij}$

(19) immediately follows from (4) & (18)

Only (20) remains to be shown. ~~Assume~~ (21), claim: (22)

3 cases, Case 1 : $m \notin S_m(x)$ or $m \in S_m(x)$ & $w \geq_m w^*(x,m)$
$\geq_m w_*(x,m)$

Then (11) & (12) implies

$$\sum_{j >_m w} \underbrace{(z^*)_{mj}}_{\substack{\text{has 1 exactly}\\ \text{at } j = w^*(x,m)\\ \text{but } w^*(x,m) \not>_m w}} = \sum_{j >_m w} \underbrace{(z_*)_{mj}}_{\substack{\text{has 1 exactly at } j = w_*(x,m)\\ \text{but } w_*(x,m) \not>_m w.}} = 0 \qquad \text{—— (23)}$$

Also, by (6), $w \in S_w(x)$ & $m \leq_w m_*(x,w) \leq_w m^*(x,w)$

So, by (14) & (15),

$$\sum_{i >_w m} \underbrace{(z^*)_{iw}}_{\substack{1 \text{ if } m <_w m^*(x,w)\\ 0 \text{ if } m = m^*(x,w)}} + \underbrace{(z^*)_{mw}}_{\substack{0 \text{ if } m <_w m^*(x,w)\\ 1 \text{ if } m = m^*(x,w)}} = \underbrace{\sum_{i >_w m} (z_*)_{iw} + (z_*)_{mw}}_{\text{similarly for } m \leq_w m_*(x,w) \text{ cases.}} = 1 \qquad \text{—— (24)}$$

(23) & (24) $\Rightarrow$ (22)

Case 2 : $w \notin S_w(x)$ or $w \in S_w(x)$ & $m \geq_w m^*(x,w)$

Symmetric arguments as the previous case.

Case 3 : $m \in S_m(x)$, $w \in S_w(x)$, $\underline{w <_m w^*(x,m)}$ & $m <_w m^*(x,w)$

$\because$ $m \in S_m(x)$ & $\underline{w <_m w^*(x,m)}$, it follows that

$$\sum_{j >_m w} (z^*)_{mj} = 1$$ as $j = w^*(x,m)$ will yield $(z^*)_{mj} = 1$, others 0.

& $(z^*)_{mw} = 0$

We want to prove $\displaystyle\sum_{i >_w m} (z^*)_{iw} = 0$

It is enough to prove that $m_*(x,w) \leq_w m$

$\because$ (21) $\equiv$ (8) holds, Converse of (6) holds. If

$m_*(x,w) \geq_w m$, then $w \geq_m w^*(x,m)$ ($\because m \in S_m(x)$ holds)

$\Rightarrow \in$, hence $m_*(x,w) <_w m$ & we are done.

So, we have $\displaystyle\sum_{j >_m w} (z^*)_{mj} + \sum_{i >_w m} (z^*)_{iw} + (z^*)_{mw} = 1$ — (26)

By symmetric arguments, we can show

$$\underbrace{\sum_{j >_m w} (z_*)_{mj}}_{\overset{\shortparallel}{0}} + \underbrace{\sum_{i >_w m} (z_*)_{iw}}_{\overset{\shortparallel}{1}} + \underbrace{(z_*)_{mw}}_{\overset{\shortparallel}{0}} = 1 \quad \text{— (27)}$$

Clearly, (26) & (27) $\Rightarrow$ (22) in this final case. $\boxtimes$.