Topics in Topology
(Homework 4)
January 19, 2015

- Each question is worth 10 points.
- Due date - February 11, 2015.

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1. In the following two situations show that the set $\mathcal{I}$ is a directed set.
   (a) For a topological space X and a given (nonempty) subset K let $\mathcal{I}$ be the collection of all subsets containing K ordered by inclusion.
   (b) Let $\mathcal{I}$ be the set of all positive integers ordered by divisibility, i.e., $m \leq n$ if and only if m divides n.

   Assume the following notation for the remaining problems. The symbol $\mathcal{I}$ will denote a directed set, $\{G_i, f_{ij} \mid i, j \in \mathcal{I}, i \leq j\}$ is a diagram of abelian groups and $\mathbf{G} := \varinjlim G_i$ is the direct limit.

2. Prove that for every $i \in \mathcal{I}$ there exist homomorphisms $\phi_i \colon G_i \to \mathbf{G}$ with the following property:
   (a) For any $i \leq j$ we have $\phi_j \circ f_{ij} = \phi_i$.
   (b) Given an abelian group $\mathbf{A}$ and homomorphisms $\psi_i \colon G_i \to A$ such that $\psi_j \circ f_{ij} = \psi_i$ whenever $i \leq j$ then there exists a unique homomorphism $F \colon \mathbf{G} \to \mathbf{A}$ and $F \circ \phi_i = \psi_i$ for every i.

3. Prove that every element of $\mathbf{G}$ can be written as $\phi_i(a)$ for some $a \in G_i$. If $a \in G_i$ satisfies $\phi_i(a) = 0$ then there is a $j \geq i$ with $f_{ij} = 0$.

4. In each of the following diagram explicitly describe the direct limit.
   (a) Suppose it is given that each G_i is a finitely generated subgroup of a fixed abelian group A and all the maps f_{ij}'s are inclusion. Then explicitly describe $\mathbf{G}$.
   (b) The diagram given is $C \xleftarrow{f} A \xrightarrow{g} B$, describe the direct limit.

5. Suppose there is a $k \in \mathcal{I}$ such that $i \leq k$ for all i. Then prove that $\mathbf{G} = G_k$. More generally, say that a subset $\mathcal{J} \subset \mathcal{I}$ is cofinal if it is a directed set with the induced ordering and if for any $i \in \mathcal{I}$, there is a $j \in \mathcal{J}$ such that $i \leq j$. Then prove that

$$\varinjlim{}_{\mathcal{J}} G_j \cong \mathbf{G}.$$