Introduction to Reflection Groups

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TRIANGLE GROUP

(COURSE PROJECT)

Abstract

A triangle group is an infinite reflection group. It is realized geometrically by sequences of reflection across the sides of a triangle. There are three types of triangle group- **Euclidean, Spherical and Hyperbolic.**

Introduction

Let X be either $\mathbb{R}^2$ or S^2 or $\mathbb{H}^2$. Let $\triangle(l,m,n)$ be a triangle in X with angles $\pi/l, \pi/m, \pi/n$ and sides α, β, γ and a, b, c be the reflection across the sides α, β, γ resepectively.

Then the following relation hold

$$a^2 = b^2 = c^2 = 1$$
$$(ab)^n = (bc)^l = (ca)^m = 1$$

The triangle in X defined using the following presention

$$\Delta(l,m,n) = \{a, b, c | a^2 = b^2 = c^2 = (ab)^n = (bc)^l = (ca)^m = 1\}$$

Definition: *Let l, m, n be the positive integers ≥ 2 and define λ to be*

$$\lambda = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 \tag{1}$$

So, a triangle group $\Delta(l,m,n)$ is reflection group which is generated by the reflections of all sides of the triangle with angles $\pi/l, \pi/m, \pi/n$.

Now if
$\lambda > 0$ *then Δ is spherical,*
$\lambda < 0$ *then Δ is hyperbolic and*
$\lambda = 0$ *then Δ is euclidean.*

Classification

1 Euclidean Space

We have for the $\mathbb{R}^2$

$$\lambda = 0$$

Hence putting it in (1) we get:

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$$

Lemma 1.1 *$\triangle(l,m,n)$ is isomorphic to $\triangle(m,n,l)$ i.e. $\triangle(l,m,n) \cong \triangle(m,n,l)$*

Proof. We have,

$$\Delta(l,m,n) = \{a,b,c|a^2 = b^2 = c^2 = (ab)^n = (bc)^l = (ca)^m = 1\}$$
$$\Delta(m,n,l) = \{x,y,z|x^2 = y^2 = z^2 = (xy)^l = (yz)^m = (zx)^n = 1\}$$

Define a map,

$$\phi : \triangle(l,m,n) \to \triangle(m,n,l)$$

such that

$$a \mapsto x$$
$$b \mapsto y$$
$$c \mapsto z$$

let $u, v \in\triangle(l,m,n)$. So we can take $u = a^i b^j c^k$ and $w = a^p b^q c^r$. Now,

$$\begin{aligned}
\phi(uw) &= \phi(a^i b^j c^k a^p b^q c^r) \\
&= \phi(a^{i+p} b^{j+q} c^{k+r}) \\
&= \phi^{i+p}(a)\phi^{j+q}(b)\phi^{k+r}(c) \\
&= x^{i+p} y^{j+q} z^{k+r} \\
&= (x^i y^j z^k)(x^p y^q z^r) \\
&= \phi(u)\phi(v)
\end{aligned}$$

So ϕ is a homomorphism. ϕ is also surjective and injective. Hence ϕ is isomorphism. Therefore $\triangle(l,m,n)$ and $\triangle(m,n,l)$ is isomorphic.

□

Theorem 1.2 *Only values of the triple (l,m,n) can take are: $(2,3,6), (2,4,4)$ and $(3,3,3)$*

Proof. As l, m, n are positive integers≥ 2 and in $\mathbb{R}^2$

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$$

Thus (l,m,n) must take the values: (2,3,6), (2,4,4), (3,3,3).

Since we have proved that

$$\triangle(l,m,n) \cong \triangle(m,n,l) \cong \triangle(n,m,l)$$

$(2,3,6),(6,3,2),(3,6,2),(2,6,3)$ etc are all similar. Actually they are just the reflection of each other. Similar argument for the $(2,4,4)$ and $(3,3,3)$ cases. Thus $(2,3,6),(2,4,4),(3,3,3)$ is the complement list of (l,m,n).

- **Describe the diagram of these groups:**

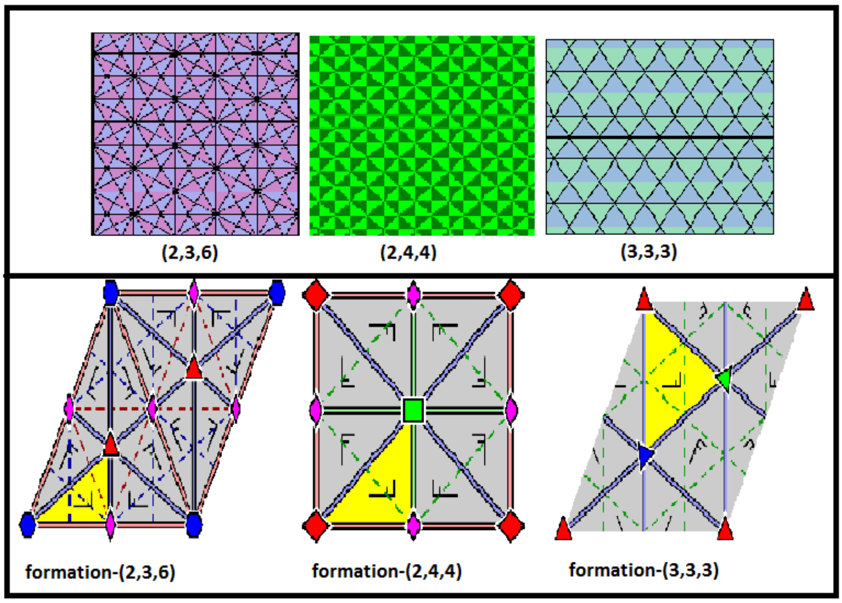


Figure 1: Euclidean triangle groups

2 Spherical Case

2.1 Postulates of Spherical Geometry

- A sphere is a subset of $\mathbb{R}^3$ given by

$$S^2 := \left\{x \in \mathbb{R}^3 \,|\|x\| = a\right\}, a > 0$$

- The radius of a sphere is the length of a drawn from the center to the surface.

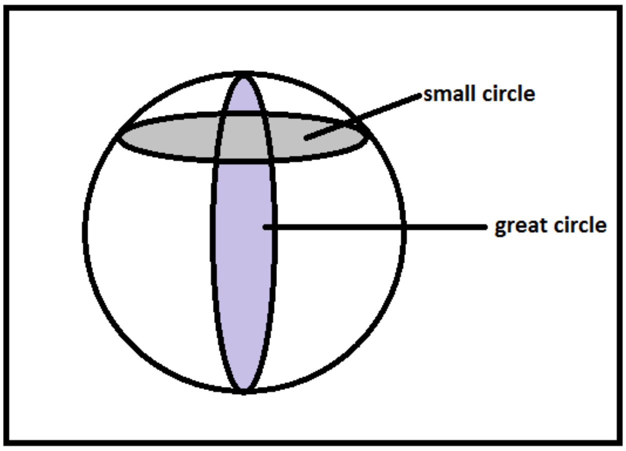


Figure 2: great and small circle

- The intersection of a sphere with a plane passing through the center is called **great circle** and its intersection with any other plane, a **small circle**.
- The points of intersection of two great circles are called **antipode**. Acctually, antipodal points are pairs of points diametrically lie on the sphere.
- The shortest path from point A to point B on a sphere is given by the shorter arc of the great circle passing through A and B is called **geodesic**. If A and B are antipodal points (like the North pole and the South pole), then there are infinitely many shortest paths between them.
- The angle made by the arcs of two great circles is called a **spherical angel**, and is to be regarded as the same with the angel between the planes of circles.Acctually, The spherical angle formed by two intersecting arcs of great circles is equal to the angle between the tangent lines to their common point.

2.2 Spherical triangles

When the arcs of three great circles intersect on the surface of a sphere, the lines enclose an area known as a **spherical triangle**. spherical triangle are distinguished as *right-angled,isosceles,equilateral* etc. same way as plane triangles.

Theorem 2.2.1 *Let AA' be a spherical segment formed by two geodesics. Let α be the angle between them. Then the area of the segment is equal to $2\alpha R^2$, where R is the radius of the sphere.*

Proof. Clearly, the area of the segment (we denote it Sα) is proportional to the area of the sphere and is proportional to the angle α. So $S\alpha$=c$\alpha(4\pi R^2)$. To find coefficient c let us notice that when $\alpha = \pi$ we should get the area of the semi-sphere, i.e. $2\pi R^2$. Therefore, $2\pi R^2 = c\pi(4\pi R^2)$, so $c = 1/2\pi$ and $S\alpha$=$2\alpha R^2$. □

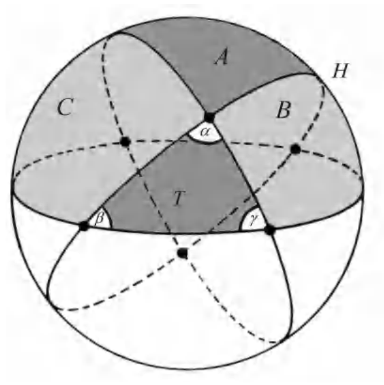


Figure 3

Lemma 2.2.2 *Let ABC be a spherical triangle with angles α, β, γ. Then the area of the triangle is $(\alpha + \beta + \gamma - \pi)R^2$, where R is the radius of the sphere.*

Proof. Let ABC be a spherical triangle with angles α, β and γ. If we continue the sides of the triangle they will meet at the other three points A', B' and C' opposite to A, B and C respectively. The triangle $\Delta A'B'C'$ is opposite to triangle ΔABC,so it has the same area as ΔABC. The rest of the surface of the sphere is covered by three spherical segments: $ABA'C'$ with angle $\pi - \alpha$, $BCB'A'$ with angle $\pi - \beta$, and $CAC'B'$ with angle $\pi - \gamma$. These segments are not overlapping, so we can write:

$$S_{ABC} + S_{A'B'C'} + S_{\pi-\alpha} + S_{\pi-\beta} + S_{\pi-\gamma} = 4\pi R^2$$

Since $S_{ABC} = S_{A'B'C'}$, using Theorem 2.2.1 we get:

$$2S_{ABC} = 4\pi R^2 - 2(\pi - \alpha)R^2 - 2(\pi - \beta)R^2 - 2(\pi - \gamma)R^2$$

Finally,

$$S_{ABC} = (\alpha + \beta + \gamma - \pi)R^2$$

□

Theorem 2.2.3 *In spherical geometry sum of the angles of any triangle is greater than π.*

Proof. Let ABC be a spherical triangle with angles $\alpha, \beta \& \gamma$. Then

$$\alpha + \beta + \gamma = \pi + S_{ABC}/R^2$$

hence

$$\alpha + \beta + \gamma = \pi + S_{ABC}/R^2 > \pi \Rightarrow \alpha + \beta + \gamma > \pi$$

where S_{ABC} is the area of the triangle and R is the radius of the sphere.

□

2.3 Spherical Triangle Group

We have in spherical case $\lambda > 0$. Substituting it in (1) we get:

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$$

So upto permutations the triples (l, m, n) of positive integers ≥ 2 are respectively $(2, 2, n \geq 2), (2, 3, 3), (2, 3, 4), (2, 3, 5)$.

In geometry, spherical triangles are called as **Schwarz triangle**. It can be used to tile a sphere,through the re ections in its edges. A Schwarz triangle is represented by three rational numbers (p q r) each representing the angle at a vertex. ”2” means a right triangle.

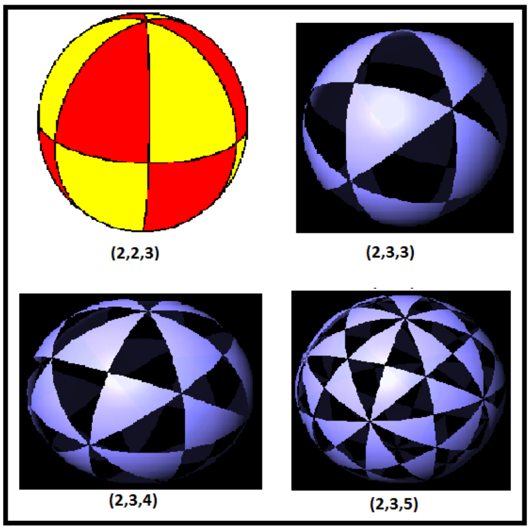


Figure 4: Spherical triangle groups

2.4 Graphical Representation

A Schwarz triangle is represented graphically by a triangular graph. Each node represents an edge (mirror) of the Schwarz triangle. Each edge is labeled by a rational value corresponding to the reflection order, being π/vertex angle.

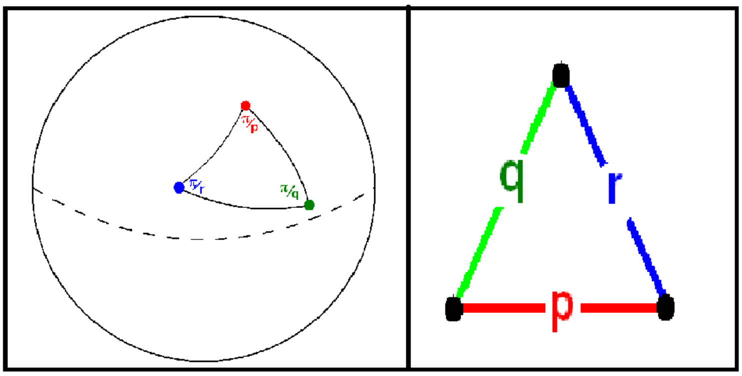


Figure 5: Schwarz triangle

Order 2 edges represents perpendicular mirrors which can be ignored in this diagram. The Coxeter-Dynkin diagram represents this triangular graph with order-2 edges hidden.

A Coxeter group can be used for a simpler notation, as (p q r) for cyclic graphs, and (p q 2) = [p,q] for (right triangles).

2.4.1 Describe the graphical representation of these group in detail:

(i) Dihedral Symmetry:

Notation: [p,2] or (p 2 2)

Fundamental domain:

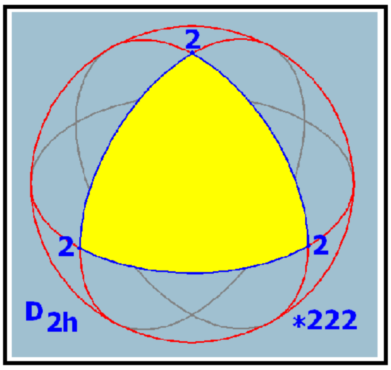


Figure 6: D_{2h}

Coxeter dynkin diagram:

Coxeter group: $A_1 \times I_2(n)$

(ii) Tetrahedral Symmetry:

Notation: [3,3] or (2 3 3)

Fundamental domain:

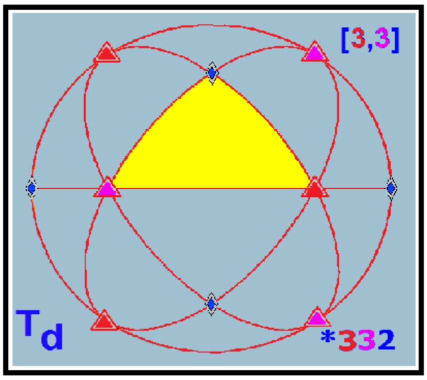


Figure 7: T_d

Coxeter dynkin diagram:

Coxeter group: A_3

(iii) Octahedral Symmetry:

Notation: [3,4] or (2 3 4)

Fundamental domain:

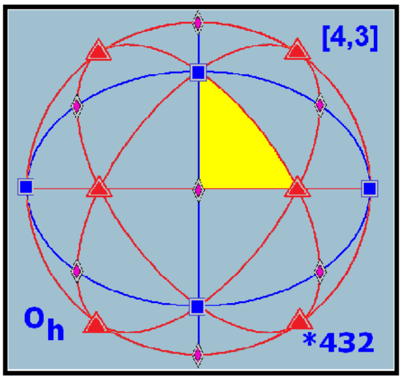
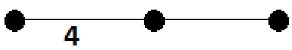


Figure 8: O_h

Coxeter dynkin diagram:



Coxeter group: BC_3

(iv) Icosahedral Symmetry:

Notation: [3,5] or (2 3 5)

Fundamental domain:

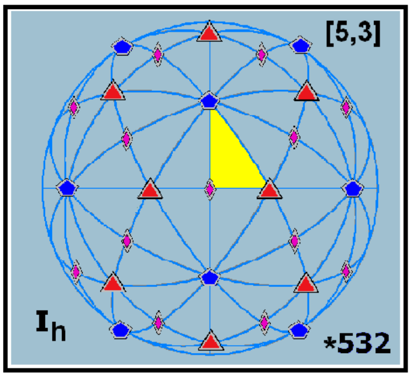
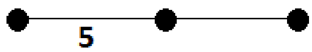


Figure 9: I_h

Coxeter dynkin diagram:



Coxeter group: H_3

2.4.2 Also describe the graphical diagram of Euclidean triangle groups:

(i) (2 4 4)-Euclidean group

Fundamental domain:

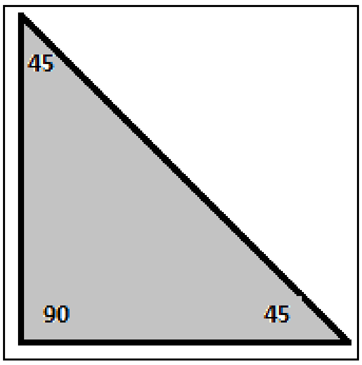
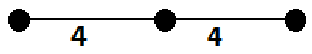


Figure 10: (2 4 4)

Coxeter dynkin diagram:



Coxeter group: $\tilde{C}_2$

(ii) (2 6 3)-Euclidean group

Fundamental domain:

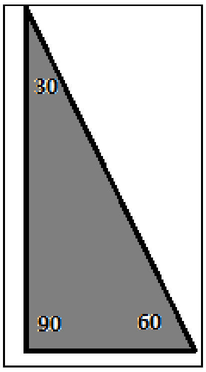
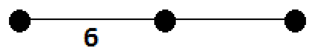


Figure 11: (2 6 3)

Coxeter dynkin diagram:



Coxeter group: $\tilde{G}_2$

(iii) (3 3 3)-Euclidean group

Fundamental domain:

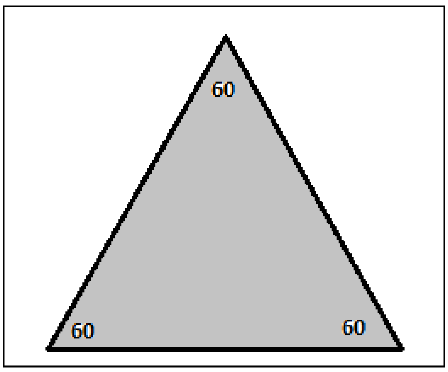
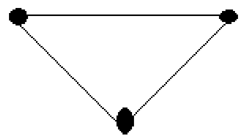


Figure 12: (3 3 3)

Coxeter dynkin diagram:



Coxeter group: $\tilde{A}_2$

3 Hyperbolic Case

We have for the $\mathbb{H}^2$

$$\lambda < 0$$

So from equtaion (1) we get

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$$

3.1 Describe the diagram of some Hyberbolic triangle groups:

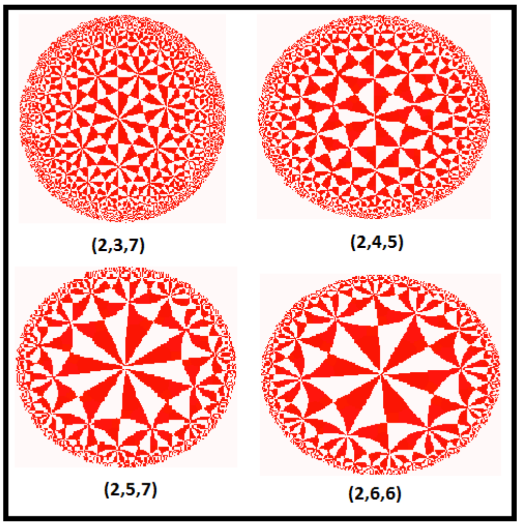


Figure 13: Hyperbolic triangle groups

References

[1] Wilhelm Magnus, Noneuclidean Tesselations and Their Groups. Academic Press New York and London, 1974.

[2] Alexandre V. Borovik, Anna Borovik, Mirror and Reflection: *The Geometry of Finite Reflection Groups*. Springer-Universitext, 2010.

[3] http://en.wikipedia.org/wiki/Triangle-group.