Research Statement

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I enjoy working on mathematical problems that syncretize ideas from topology, combinatorics and algebra. The theory of arrangements (of hyperplanes, subspaces, points etc.) and configuration spaces are some areas in mathematics that lie at an intersection of the fields I like. My doctoral work involved generalizing some results about hyperplane arrangements to smooth manifolds.

1 Doctoral work

My thesis project was inspired by the work of Deligne on Artin groups [4], the work of Salvetti on hyperplane complements [10] and on finite-type Artin groups [9], Mike Davis' work on manifold reflection groups [2, Chapter 10], Delucchi's work on the covers of the arrangement complement [5] and finally, Zaslavsky's work on topological dissection problems [11, 12].

I introduced the notion of an *arrangement of submanifolds* in a real, smooth manifold. This is a finite collection of codimension-1 submanifolds such that all the intersections are hyperplanes-like and the stratification induces the structure of a totally normal cellularly stratified space (intuitively, the strata are polytope-like). Simply put, the local picture is of hyperplane arrangements however, the global scene could be quite different.

Let X be an l-manifold and $\mathscr{A} = \{N_1, \ldots, N_k\}$ be a submanifold arrangement. Let TX denote the tangent bundle of X. The *tangent bundle complement*, denoted $M(\mathscr{A})$, is defined as $TX \setminus \bigcup_{i=1}^{k} TN_i$. The combinatorial information regarding these arrangements is encoded in the the *intersection poset* and the *face category*. The main theme of investigation was:

To what extent do the face category and the intersection poset determine the topology of the tangent bundle complement?

The above theme was pursued by trying to generalize well-known theorems in the area of hyperplane arrangements to the context of submanifold arrangements. For example, counting the number of connected components in the complement of the arrangement (Section 2.1), generalization of the Salvetti complex to manifold reflection arrangements (Section 2.5) (I should note here that the proof of Salvetti's result in full generality is in progress) etc. Since the thesis contained many different directions the results that were proved (or announced) were then extracted in separate articles. Remaining sections contain a summary of the results that were proved and a discussion about future (or ongoing) projects. In case of completed projects a reference to the preprint is provided at the end of relevant subsections (for the reader's benefit there is a list of publications at the end, Section 3.8).
The thesis is available for download at http://ir.lib.uwo.ca/etd/154/

2 Post-doctoral work

This section describes projects completed during the post-doctoral years.

2.1 Generalization of Zaslavsky's theorem

Consider the complement of the arrangement inside the manifold, which is disconnected. One is interested in counting the number of connected components (chambers). The counting is done using the Möbius function of the intersection poset and the compactly supported cohomology of the intersections.

Theorem 2.1. *Let $\mathscr{A}$ be an arrangement of submanifolds with the intersection poset L and let* $\mathrm{Poin}_c(Y,t)$ *denote the Poincaré polynomial with compact supports of a topological space Y.The function $\nu\colon L \to \mathbb{Z}[t]$ defined by $\nu(A) =$* $\mathrm{Poin}_c(A,t)$, $\forall A \in L$ *extends uniquely to a measure on the distributive lattice generated by L. Then the number of chambers is given by*

$$|\mathscr{C}(\mathscr{A})| = (-1)^l \sum_{Y\in L} \mu(X,Y)\mathrm{Poin}_c(Y,-1).$$

Reference: On a generalization of Zaslavsky's theorem for hyperplane arrangements, arXiv:1111.1251.
(Annals of Combinatorics, Springer Basel, 2014, 18, 35-55)

2.2 Arrangements of pseudohyperplanes

The oriented matroids are intimately connected to hyperplane arrangements. An important landmark in the theory of oriented matroids is the *Folkman-Lawrence Topological Representation Theorem* [8]. The theorem states that there is a one-to-one correspondence between oriented matroids of rank n and arrangements of pseudospheres in S^{n-1}. An arrangement of pseudospheres is a collection of mildly deformed subspheres of the unit sphere. Under this correspondence realizable oriented matroids correspond to arrangements of central hyperplanes.

A missing piece in this setting was an analogue of the complexification of a (non-stretchable) pseudo-arrangement. We first suggested to look at pseudohyperplanes instead of pseudospheres. Then using the theory of microbundles we introduced a connected, $2n$-dimensional topological space naturally associated with a pseudo-arrangement that has the homotopy type of the Salvetti complex. In case the arrangement is stretchable this space is homeomorphic to the complexified complement.

Reference: On arrangements of pseudohyperplanes, arXiv:1201.1306
(to appear in Proceedings of Mathematical Sciences).

2.3 Arrangements of spheres and projective spaces

This project aims at studying arrangements of spheres in details. The main result of the paper is an explicit description of the homotopy type of the tangent bundle complement.

Theorem 2.2. *Let $\mathscr{A}$ be an arrangement of hyperspheres in S^l such that there exists a generic equator which cuts the sphere into two open disks D^l_+, D^l_- and $\mathscr{A}_+ := \mathscr{A}|D^l_+, \mathscr{A}_- := \mathscr{A}|D^l_-$ are isomorphic pseudo-hyperplane arrangements. Let m be the number of chambers of $\mathscr{A}_-$. Then,*

$$M(\mathscr{A}) \simeq (\bigvee_m S^l) \vee Sal(\mathscr{A}_+).$$

Motivations to study these arrangements come from the the fact that the fundamental group of the complement has potential to shed some more light on the class of arrangement groups. All the results concerning spheres are also extended to the arrangements of projective spaces. Most of this work was completed during the post-doctoral stay at Northeastern University.

Reference : Arrangements of spheres and projective spaces, arXiv:1201.2193
(to appear in Rocky Mountain Journal of Mathematics).

2.4 Cohomology algebra of toric complements

Toric arrangements have recently gained popularity because of their interactions with other ares of mathematics. For example, see [7] for a combinatorial study of arrangements on a real torus. De Concini and Procesi in [3] have explored various aspects of toric arrangements related to polytopes, approximation theory, partition functions etc.

Definition 2.3. Let $(\mathbb{C}^*)^l$ be the complex l-torus. A *toric arrangement* is a finite collection $\mathscr{A} = \{H_1,\ldots,H_n\}$ of finitely many codimension-1 complex subtori in $(\mathbb{C}^*)^l$.

In a joint project with Kavita Sutar we use the Gysin sequence to identify a class of toric arrangements, called *deletion-restriction* (DR) type, for which the complement is formal in the sense of Sullivan and its cohomology algebra is generated in degree 1. This work extends a theorem of De Concini and Procesi in the sense that the unimodular arrangements they considered are of DR-type.
Reference : Deletion-restriction for toric arrangements (with K. Sutar), arXiv:1406.0302
(to appear in Journal of Ramanujan Mathematical Society).

2.5 REFLECTIONS ON MANIFOLDS

The most important work that initiated the study of the complexified complement of a real hyperplane arrangement is that of Deligne [4]. Aim of this project was to generalize the seminal theorem Salvetti that appeared in [9]. The following work also appeared in the masters' thesis of Ronno Das.

Definition 2.4. Let X be a real, connected, smooth manifold of finite dimension l. A *reflection of* X is a self-diffeomorphism s such that $s^2 = 1$, the fixed set X_s is a smooth, codimension-1 submanifold and $X \setminus X_s$ has 2 distinct components.

Let $W(X,n)$ denote the group generated by n distinct dissecting reflections of X and let N_i denote the subset fixed by i-th reflection. An important theorem due to Vinberg states that the group $W(X,n)$ has a Coxeter presentation (see [2, Chapter 10]). It is not hard to see that the collection of these N_i's is a submanifold arrangement and there is a fixed point free, proper action of $W(X,n)$ on the tangent bundle complement. We show that the combinatorial data of the Coxeter transformation group can be used to construct a cell complex homotopy equivalent to the tangent bundle complement and that this homotopy equivalence is equivariant. In particular our results generalize the work of Salvetti [9] on finite-type Artin groups and of Charney-Davis [1] on arbitrary Artin groups.
Reference : Coxeter transformation groups and reflection arrangements in smooth manifolds (with R. Das) arXiv:1408.3921 (to appear in Journal of Homotopy and Related Structures).

2.6 FACE ENUMERATION PROBLEMS

A real toric arrangement is a finite collection of codimension-1 subtori in $\mathbb{R}^n/\mathbb{Z}^n$. These subtori stratify the ambient torus into faces of various dimensions. Let f_i denote the number of i-dimensional faces; these *face numbers* satisfy the Euler relation $\sum_i (-1)^i f_i = 0$. However not all tuples of natural numbers satisfying this relation arise as face numbers of some toric arrangement. In the joint work with Karthik Chandrashekhar we give complete characterization of face numbers of arrangements in a 2-dimensional torus. In particular we have proved that the convex hull of these face numbers is a positive cone in $\mathbb{R}^2$.
Reference : Face enumeration for line arrangements in a 2-torus (with K. Chandrashekhar), arXiv:1404.1665.

3 WORK IN PROGRESS AND FUTURE PROJECTS

This section lists projects in progress as well as problems I plan to pursue in near future.

3.1 DUALITY PROPERTIES OF THE TANGENT BUNDLE COMPLEMENT

This is joint work with Ronno Das. We consider the action of a finite reflection group on the tangent bundle of the unit sphere. The action fixes a union of tangent bundles of fixed sub-spheres. Using the spectral sequence arguments developed by Mike Davis and his co-authors we show that the twisted cohomology of the complement vanishes everywhere except in the top dimension.

3.2 COMPLEMENTS OF AFFINE CURVES

This is joint work with Anwesh Ray. Here we consider the complement of finitely many non-singular (affine) curves in $\mathbb{C}^2$. To be precise we consider a finite collection of open Riemann surfaces that intersect (locally) like (complex) lines. We have calculated the the cohomology algebra of this space and expressed the relations in terms of the intersection data. In particular, this algebra has generators in degree 2.

Using Leray's residue theory we also show that the complement is formal in the sense of Sullivan. This observation seems interesting for the following reason - Recently, Clement Dupont [6] has proved that if the mixed Hodge structure on the k-th cohomology group of every codimension k intersection (for all k's) of a hypersurface arrangement is pure then the complement of this arrangement is formal. Our work suggests that Dupont's condition is only sufficient, since the first cohomology of an open Riemann surface of positive genus has classes of weights 1 and 2. Hence we provide an example of hypersurface arrangements for which the complement is formal however the mixed Hodge structure on the cohomologies of the intersections is not pure.

3.3 Dissection and enumeration problems

There are two parts of this project (joint with K. Chandrashekhar). In the first we are trying to characterize face vectors of a simplicial toric arrangements. In particular, we have an analogue of Dehn-Sommerville relations for arrangements that define a Δ-complex structure. The second part is regarding arrangements of geodesics in the hyperbolic plane and surfaces. We currently have a partial characterization of face numbers.

3.4 Toric arrangements

Determining a presentation for $H^*(M(\mathscr{A}),\mathbb{C})$ when $\mathscr{A}$ is a DR-type toric arrangement is in progress. This will also help prove that cohomology is torsion free. An interesting sub-class is that of arrangements defined by affine root systems. It seems plausible that study of these arrangements might shed some light on containment relations between various affine Artin groups.

3.5 Reflections on manifolds

The aim of this project is to identify submanifold arrangements for which $\pi_1(M(\mathscr{A}))$ has solvable word problem (joint with R. Das). Moreover we are also trying to characterize the cases in which the tangent bundle complement is $K(\pi,1)$ and study homological finiteness properties of $\pi_1(M(\mathscr{A}))$.

3.6 Combinatorics of faces of an arrangement

We first state an interesting result:

Theorem 3.1 (Schneider 87, Varchenko 88, Fukuda et. al. 91)**.** *Let $\mathscr{A}$ be a central arrangement of hyperplanes in $\mathbb{R}^l$. The arrangement determines a cellular decomposition of the unit sphere. Then the average number of j-faces of a k-face of the unit sphere is less than the number of j-faces of the k-cube.*

The above theorem raises following probabilistic questions for affine arrangements: to compute the probability that a randomly selected k-face is bounded; and to compute the probability that a bounded k-face is combinatorially equivalent to a k-cube or a k-simplex.

3.7 Covering spaces of the tangent bundle complement

Building on the work of Salvetti, Paris and Delucchi in his thesis described the Salvetti complex (for real hyperplane arrangements) and its connected covers as homotopy colimits of diagrams, see [5, Chapter 4]. Our plan is to generalize his results to the tangent bundle complement using homotopy colimits in the context of acyclic categories.

3.8 Cohomology groups of the tangent bundle complement

One of the benefits of using the language of homotopy colimits is that some machinery from homotopy theory can be applied. For example, if a topological space is expressed as a homotopy colimit then its cohomology can be computed using the Bousfield-Kan spectral sequence. Since the Salvetti complex is expressed as a homotopy colimit, we can apply the Bousfield-Kan spectral sequence. An explicit description of the differentials on the E_1 page is obtained, with the help of which terms on the E_2 page are calculated. Our aim is to complete these calculations.

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LIST OF RESEARCH ARTICLES

- Published
  1. *On a generalization of Zaslavsky's theorem for hyperplane arrangements*, Annals of Combinatorics, Springer Basel, 2014, 18, 35-55, arXiv:1111.1251.
- Accepted for publication
  1. *Arrangements of spheres and projective spaces*, Rocky Mountain Journal of Mathematics, arXiv:1201.2193.
  2. *Deletion-restriction for toric arrangements* (with K. Sutar), Journal of Ramanujan Mathematical Society, arXiv:1406.0302.
  3. *On arrangements of pseudohyperplanes*, Proceedings Mathematical Sciences, arXiv:1201.1306.
  4. *Coxeter transformation groups and reflection arrangements in smooth manifolds* (with R. Das), Journal of Homotopy and Related Structures, arXiv:1408.3921.
- arXiV preprints
  1. *Arrangements of submanifolds and the tangent bundle complement*, arXiv:1110.1520.
  2. *Face enumeration for line arrangements in a 2-torus* (with K. Chandrashekhar), arXiv:1404.1665.
- Preprints in preparation
  1. *Topological aspects of reflection arrangements in spheres* (with R. Das).
  2. *On the topology of the complement of affine curves in* $\mathbb{C}^2$ (with A. Ray).
  3. *Salvetti-type diagram models for tangent bundle complements.*