Quantum Mechanics III: Midsemester examination

Total: 35 marks Time: 3hrs.

(1) Consider a Lagrangian $L = \frac{1}{2}\int d^3x\ (|\partial_t\phi|^2 - |\nabla\phi|^2)$ (the $|A|^2$ meaning A^*A) describing a (relativistic) complex scalar field, with the usual (wave) equation of motion $\partial_t^2\phi - \nabla^2\phi = 0$. "Lightcone" coordinates are defined as $x^+ = t + x_1,\ x^- = t - x_1$. Now consider field configurations of the form $\phi = e^{ip_-x^-}\chi(x^+, x_2, x_3)$, with lightcone momentum p^-, and χ being an arbitrary function of the remaining 3 coordinates. Express the t, x_1-derivatives in terms of $\partial_{x^\pm}$. Expand the Lagrangian about such a background and show that the resulting Lagrangian resembles that for the nonrelativistic Schrodinger equation in the $\{x^+, x_2, x_3\}$-space. As a check, show that the resulting Euler Lagrange equation is in fact the Schrodinger equation, with p_- appearing as a parameter resembling mass. [10 mks.]

(2) Consider the fermion hopping model (no potential) with Hamiltonian $H = H_K = -t\sum_j (c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1})$, the c_j being fermion operators satisfying the algebra $\{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$.

(i) Treating H_K as the zeroth order Hamiltonian, diagonalize it by going to momentum basis: find the resulting energy ϵ_k, where H_K now has the form $H_K = \sum_k \epsilon_k \tilde{c}_k^\dagger \tilde{c}_k$. [2 mks.]

(ii) Consider an interaction potential between nearest neighbour sites of the form $H_U = U\sum_j c_{j+1}^\dagger c_{j+1} c_j^\dagger c_j$. Recast this in the momentum basis in terms of the $\tilde{c}_k$. [3 mks.]

(3) Consider a polaron-like Hamiltonian $H^{int} = ic\sum_k \frac{1}{k}(a_k^\dagger e^{-ik\cdot X} - a_k e^{ik\cdot X})$, c being the interaction constant (we're working in units where $m_e, \omega_k = 1$), where $a_k^\dagger$ are phonon creation operators. Consider a state $|P_1, P_2\rangle$ with 2 electrons with momenta P_1, P_2, which has energy $E = \frac{P_1^2}{2} + \frac{P_2^2}{2}$ in these units. Treat the 2-electron state as containing two identical particles, writing it as a(n antisymmetric) product of two 1-electron states. Calculate the corrections to the energy of this state in 2nd order perturbation theory. You will first need to find the matrix element for H_{0m}^{int}, where $|m\rangle$ is an intermediate state. Can you guess diagrams that represent this process? [8 mks.]

(4) Consider the Jordan-Wigner transformation on a system with just 2 lattice sites:

$$\hat{\sigma}_1^+ = c_1, \quad \hat{\sigma}_1^- = c_1^\dagger, \quad \hat{\sigma}_2^+ = (1 - 2c_1^\dagger c_1)c_2, \quad \hat{\sigma}_2^- = (1 - 2c_1^\dagger c_1)c_2^\dagger,$$

where the fermion operators satisfy the algebra ($\{,\}$ being anticommutators)

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0.$$

Calculate explicitly the commutators $[\hat{\sigma}_1^+, \hat{\sigma}_1^-]$, $[\hat{\sigma}_1^+, \hat{\sigma}_2^+]$, $[\hat{\sigma}_2^+, \hat{\sigma}_2^-]$, $[\hat{\sigma}_1^z, \hat{\sigma}_2^+]$, $[\hat{\sigma}_2^z, \hat{\sigma}_2^+]$, with $\hat{\sigma}_i^z = 1 - 2c_i^\dagger c_i$, and show that they satisfy the usual Pauli spin algebra commutation relations $[\hat{\sigma}_i^+, \hat{\sigma}_j^-] = \delta_{ij}\hat{\sigma}_i^z$, $[\hat{\sigma}_i^z, \hat{\sigma}_j^\pm] = \pm 2\delta_{ij}\hat{\sigma}_i^\pm$. [12 mks.]