Cosmologies with null Big-Bang singularities and their gauge theory duals

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- AdS/CFT with cosmological singularities
- Their holographic gauge theory duals
- Vanishing conformal anomaly; conformal dressing…
- Tilde variables. Towards gauge theory nonsingularity

Time dependence in string theory

- Usually broken spacetime supersymmetry $\Rightarrow$ time dependence. Metastable/unstable stringy vacua (tachyon dynamics etc).
- Time in string theory ? Beginning/end of time (Big Bang/Crunch) in string theory models ? Good approximations to our Universe ?
- General Relativity breaks down at singularities: so want "stringy" description. Smooth quantum (stringy) completion of classical spacetime geometry ?

Nice stringy playground: AdS/CFT. Bulk string theory on $AdS_5 \times S^5$ with metric $ds^2 = \frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) + ds^2_{S^5}$, (Poincare coords) with 5-form field strength, and dilaton $\Phi = const$, dual to boundary $d = 4\,\mathcal{N}{=}4$ (large N) $SU(N)$ Super Yang-Mills theory.

Time-dependent deformations: cosmologies

Deformations of AdS/CFT: either growing towards boundary (non-normalizable) or subleading at boundary (normalizable).

Turn on non-normalizable deformations for the metric and dilaton:

$$
\begin{aligned}
ds^2 &= \frac{1}{z^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dz^2) + ds^2_{S^5} \,, \\
\Phi &= \Phi(x^\mu) \,, \qquad \text{also nontrivial } 5-\text{form} \,.
\end{aligned}
$$

This is a solution in string theory if

$$
\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi \,, \qquad \frac{1}{\sqrt{\tilde{g}}}\,\partial_\mu(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0 \,,
$$

$i.e.$ if it is a solution to a 4-dim Einstein-dilaton system.

The general solutions

Harmonic function $Z = Z(x^m)$ (and appropriate 5-form)

$$ds^2 = Z^{-1/2}\tilde{g}_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}g_{mn}dx^m dx^n \,, \quad \Phi = \Phi(x^\mu).$$

$g_{mn}(x^m)$ is Ricci flat, and $\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}(x^\mu)$. $[\mu = 0123, m = 4\ldots 9.]$
Dilaton satisfies its EOM. And the Type IIB sugra EOM is

$$R_{MN} = \frac{1}{6}F_{MABCD}F_N{}^{ABCD} + \frac{1}{2}\partial_M\Phi\partial_N\Phi.$$

If $\tilde{g}_{\mu\nu}$ were flat, and dilaton constant, then such solutions are well-known ($e.g.$ coincident D3-branes at a conical singularity with base space $g^{\perp}_{mn}$). 5-form effectively acts as 5D cosmological constant. New contribution to R_{MN} is $\tilde{R}_{\mu\nu}$, so

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi.$$

Time-dependent/Null cosmologies

Spacelike: Consider $\tilde{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \sum_{i=1}^3 t^{(2p_i)} dx^i dx^i$ and $e^\Phi = t^\alpha$. We obtain solutions generalizing Kasner-like cosmologies if $\sum_i p_i = 1$, $\frac{\alpha^2}{2} = 1 - \sum_i p_i^2$, from the R_{tt}, R_{ii}, Φ EOM. Restrictive. Can be generalized to other $\tilde{g}_{\mu\nu}$.

Contain *spacelike* cosmological singularities.

Null: Consider $\tilde{g}_{\mu\nu} dx^\mu dx^\nu = e^{f(X^+)}(-2dX^+ dX^- + dx^i dx^i)$, and $\Phi = \Phi(X^+)$, where $X^+ =$ lightlike coord. These are solutions if

$$\frac{1}{2}(\partial_+ \phi)^2 = \tilde{R}_{++} = \frac{1}{2}(f')^2 - f'' \, . \qquad \left(f' = \frac{\partial f}{\partial X^+} \right)$$

Dilaton EOM $\partial_\mu (\sqrt{-\tilde{g}}\, \tilde{g}^{\mu\nu} \partial_\nu \Phi) = 0$ automatically satisfied since $\Phi = \Phi(X^+) \Rightarrow$ infinite family of solutions parametrized by $\Phi(X^+)$. Null singularities here. 8 lightcone supercharges preserved.

Prototypical example

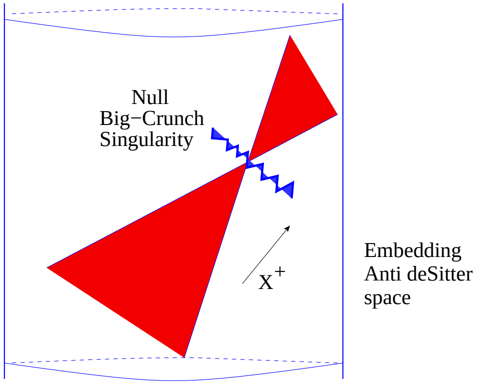
Consider $e^f = \tanh^2 X^+$

$$d\tilde{s}^2 = \tanh^2 X^+(-2dX^+dX^- + dx_2^2 + dx_3^2),$$
$$e^\Phi = g_s \left|\tanh\frac{X^+}{2}\right|^{\sqrt{8}}.$$

Far past/future: $AdS_5 \times S^5$ with dilaton constant. As $X^+ \to 0$, singularity (at finite affine time) as $e^f \to 0$, with $R_{++} = \frac{4}{\sinh^2 X^+}$.
EOM satisfied for $X^+ \neq 0$ and continuous at $X^+ = 0$.
With g_s small, dilaton can be made small everywhere.
Note: only solution with everywhere constant dilaton is $e^f = \frac{1}{(X^+)^2}$, which is flat space $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ using

$$x_i = X^+Y_i, \;\; X^- = Y^- - X^+(Y_2^2 + Y_3^2), \;\; X^+ = -\frac{1}{Y^+}.$$

Nature of null Big-Bang/Crunch singularity



These contain null Big-Bang (Crunch) cosmological singularities when the transverse space shrinks as $e^f \to 0$, at say $X^+ = 0$. Then curvature along infalling null geodesics ξ^μ diverges

$$\tilde{R}_{ab}\xi^a\xi^b = \tilde{R}_{++}e^{-2f} \to \infty.$$

Diverging compressional tidal forces along infalling null geodesic congruence. Other invariants $R,\ R_{AB}R^{AB}$ etc as in AdS_5.
Consider $e^f = \tanh^2 X^+$ as limit of $e^f = (|\tanh X^+| + \epsilon)^2$. Then $e^\Phi \sim g_s(\epsilon)^{\sqrt{8}}$. Curvature, affine parameter: continuous, nonsingular.

The gauge theory duals

- Conjecture: Type IIB string theory on these backgrounds is dual to $\mathcal{N}=4$ $d=4$ SYM on a base space $\tilde{g}_{\mu\nu}$ with a time dependent gauge coupling $g^2_{YM} = e^{\Phi}$.
  Natural extension of AdS/CFT for small perturbations $\delta\Phi,\ \delta g_{\mu\nu}$:
  $$S = \int d^4x\, [\frac{\delta\Phi(x^\mu)}{g^2_{\rm YM}} TrF^2 + \delta g_{\mu\nu} T^{\mu\nu}],$$
  $i.e.$ the dual is $\mathcal{N}=4$ SYM theory with these sources turned on.
- Analyzing the D-probe DBI action corroborates this. Imagine building up this spacetime by stacking D3-branes in a background $ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu + dx^m dx_m$ and dilaton $\Phi(x^\mu)$. This gives $ds^2 = Z^{-1/2}(x)\tilde{g}_{\mu\nu} dx^\mu dx^\nu + Z^{1/2}(x) dx^m dx_m$, with dilaton and appropriate 5-form. Now take near horizon limit.

$Questions$: Vanishing absorption cross-sections (near singularity)? D-brane boundary states in time-dependent backgrounds ?

The gauge theory duals cont'd.

Reverse question: time-dependent deformations of $\mathcal{N}=4$ SYM ? Start in $\mathcal{N}=4$ vacuum in the far past; turn on time-dependent gauge coupling and a time-dependent initially flat base space. Gauge theory response ? Well-posed problem: $\Phi, \tilde{g}_{\mu\nu}$ specify gauge theory data completely.

Supergravity dual ? If the dilaton and metric are related by the (IIB) equations earlier, then a sugra dual is straightforward to identify. Nontrivial sources turned on (for operators $TrF^2, T_{\mu\nu}$) are dilaton Φ and metric $\tilde{g}_{\mu\nu}$ $\Rightarrow$ then sugra dual is this deformation of $AdS_5 \times S^5$.

Is the gauge theory $nonsingular$?
Null cosmologies special: gauge theory dual lives on base space $\tilde{g}_{\mu\nu} = e^{f(x^+)}\eta_{\mu\nu}$ conformal to flat space, and has null-time-dependent gauge coupling $g_{YM}^2 = e^{\Phi(x^+)}$.

The trace anomaly

The trace anomaly for a field theory on a curved background is

$$\begin{aligned} T_\mu{}^\mu &\propto c(C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}) - a(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2) \\ &\propto -R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}R^2. \end{aligned}$$

For $SU(N)$ $\mathcal{N}=4$ SYM, we have $c = a = \frac{N^2-1}{4}$.
In the time dependent cosmologies, both terms $\propto \frac{1}{t^4}$.

Null cases: only nonzero R_{ab} is R_{++} $\Rightarrow$ $T_\mu{}^\mu = 0$ for any $f(X^+)$.
With time-varying dilaton: any additional term in $T_\mu{}^\mu$ must be generally covariant involving dilaton derivatives, and tensors made out of the metric. These vanish since only $\partial_+\Phi$ nonzero, and no tensor with two (or more) upper $+$ components.

CFT in conformally flat bgnd

Consider the partition function (ignore time-dependent coupling)

$$Z[g_{\mu\nu}] = \int [D\varphi]_{[g_{\mu\nu}]} e^{iS[g_{\mu\nu},\varphi]},$$

S being the action over fields φ. Under metric variations $\delta g_{\mu\nu}$, $\delta Z = \int [D\varphi]_{[g_{\mu\nu}]} e^{iS[g_{\mu\nu},\varphi]} \left(i \int d^4x \sqrt{-g}\, \delta g_{\mu\nu} T^{\mu\nu}\right)$, $T^{\mu\nu}$ being the stress tensor. Thus under Weyl rescalings $g_{\mu\nu} \to e^{\delta\psi} g_{\mu\nu}$, $\delta \log Z = i\langle \int d^4x \sqrt{-g}\, T_\mu{}^\mu \delta\psi \rangle$.

Consider 1-parameter family of metrics $g_{\mu\nu} = e^{\alpha f(X^+)} \eta_{\mu\nu},\ \alpha \in [0,1]$.
Then $T_\mu{}^\mu = 0 \ \Rightarrow\ \partial_\alpha Z[g_{\mu\nu}] = 0 \ \Rightarrow\ Z[e^f \eta_{\mu\nu}] = Z[\eta_{\mu\nu}]$.
Similarly for correlation functions

$$\langle \prod_i e^{\frac{\alpha f(x_i) \Delta_i}{2}} \mathcal{O}(x_i) \rangle_{[e^f \eta_{\mu\nu}]} = \langle \prod_i \mathcal{O}(x_i) \rangle_{[\eta_{\mu\nu}]}.$$

Varying dilaton effects

For our prototypical example, far past/future state is the $\mathcal{N}=4$ SYM conformal vacuum. In general, time-varying interactions lead to particle production: null backgrounds ?

Consider conformal scalar with null-dependent interaction:

$$S = \int d^4x\sqrt{-g}\ \left[\tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{6}R\varphi^2 + J(X^+)\varphi^3\right] .$$

Lightcone quantization: modes $e^{-i(k_i x^i + k_- X^- + \frac{k_i^2}{2k_-}X^+)}$ are positive frequency ($k_+ = \frac{k_i^2}{2k_-} > 0$) w.r.t. $X^+ \ \Rightarrow\ k_- \geq 0$.

Mode expanding in the free theory gives

$$\varphi = e^{-\frac{f}{2}} \int d^2k \int_0^\infty \frac{dk_-}{\sqrt{(2\pi)^3 2k_-}}\ [a_k e^{-i(k_i x^i + k_- X^- + \frac{k_i^2}{2k_-}X^+)} + \text{c.c.}]$$

Conformal vacuum: $\quad a(k_i, k_-)|0\rangle = 0$.

Varying dilaton effects cont'd

- Vacuum of the non-interacting theory remains unchanged with a null-dependent source $\Rightarrow$ no particle production.
  Essentially, final interaction picture state
  $|s\rangle = T_+ e^{-i\int d^4x \; e^{2f(X^+)} J(X^+) \; \varphi^3} |0\rangle$ remains unchanged.
  Physically: analogous to space-varying source. X^--translation invariance means P_- conserved since source $J(X^+)$ does not break this symmetry. $Caveat$: $k_- = 0$ subtleties.

- In perturbation theory in source $J(X^+)$, correlators with interaction $\int d^4X e^{\frac{f}{2}} J(X^+) e^{\frac{3f}{2}} \varphi^3(X)$ related to free correlators, and thus to flat space $dressed$ correlators.
  Operator $\mathcal{O}(x)$ of conformal dimension Δ dressed as $e^{\frac{f\Delta}{2}} \mathcal{O}(x)$.
  For source $J(X^+)$ coupling to $\mathcal{O}(x)$, interaction damped if $J(X^+) e^{\frac{4-\Delta}{2} f(X^+)} \rightarrow 0$ as $X^+ \rightarrow 0$.

Field redefinitions: toy scalar

With varying dilaton, kinetic terms of gauge fields are nontrivial. Want redefinition to new variables with canonical kinetic terms.

$Toy\ model$: scalar field $\int d^4x\, e^{-\Phi(X^+)}(\partial\varphi)^2$. Redefine $\varphi = \epsilon(x)\tilde{\varphi}$:
$\int d^4x\, e^{-\Phi}\, \eta^{\mu\nu} \left(\epsilon^2 \partial_\mu\tilde{\varphi}\partial_\nu\tilde{\varphi} + \epsilon\partial_\mu\epsilon\partial_\nu(\tilde{\varphi}^2) + (\partial_\mu\epsilon\partial_\nu\epsilon)\tilde{\varphi}^2\right)$.
Now if $\epsilon(x) = e^{\Phi(X^+)/2}$, first term canonical kinetic term. And ϵ is null $\Rightarrow$ (i) the third term vanishes, (ii) the second term is a total derivative $\partial_+(\epsilon^2)\partial_-(\tilde{\varphi}^2) = \partial_-[\partial_+(\epsilon^2)\tilde{\varphi}^2]$, which can be dropped.

Consider $interactions$:
$-\int d^4x\, e^{-\Phi(X^+)}[(\partial\varphi)^2 - \lambda\varphi^4] \;\rightarrow\; -\int d^4x[(\partial\tilde{\varphi})^2 - \lambda e^{\Phi(X^+)}\tilde{\varphi}^4]$.

Thus $\tilde{\varphi}$-variables have canonical kinetic terms. As $X^+ \rightarrow 0$, if $e^{\Phi} \rightarrow 0$, then $\tilde{\varphi}$-interaction damped $\Rightarrow$ nonsingular S-matrix. Theory well-defined, transparent in $\tilde{\varphi}$-variables used for defining asymptotic states.

$\mathcal{N}=4$ SYM: the tilde variables

$\mathcal{N}=4$ SYM: near singularity, $e^{\Phi} \to 0$, so kinetic terms singular:
want well-defined variables $\int e^{-\Phi} \mathrm{tr} F^2 + \ldots \to \int \mathrm{tr} \tilde{F}^2 + \ldots$
For simplicity, work in lightcone gauge $A_- = 0$ (and flat metric).
Define $\tilde{A}_\mu = e^{-\Phi/2} A_\mu$. Then $S_{\rm GF} = -\frac{1}{4} \int d^4x\, e^{-\Phi}\, \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to$

$$- \int \frac{d^4x}{4} \,[\, \mathrm{Tr}(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu)^2 - 2i e^{\Phi/2} \mathrm{Tr}\{(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu)[\tilde{A}^\mu, \tilde{A}^\nu]\}$$
$$- e^{\Phi} \mathrm{Tr}([\tilde{A}_\mu, \tilde{A}_\nu])^2 - \partial_- \{(\partial_+ \Phi) \tilde{A}_i \tilde{A}^i\}\,]$$

The last term is a total derivative and does not affect the EOM.
Other interaction terms containing the dilaton:
$\int d^4x [e^{\frac{\Phi}{2}} J^{\mu a} \tilde{A}_{\mu a} + e^{\Phi} \mathrm{Tr}([\tilde{A}_\mu, \phi^\alpha][\tilde{A}^\mu, \phi^\alpha]) + e^{\Phi} \mathrm{Tr}([\phi^\alpha, \phi^\beta][\phi^\alpha, \phi^\beta])]$,
where $J^{\mu a}$ is the gauge current from scalars ϕ^α and fermions.
Note: Dilaton couples with positive powers.

$\mathcal{N}=4$ SYM tilde variables cont'd.

We have imposed $A_- = 0$ gauge: A_+ nondynamical (also therefore $\tilde{A}_+$). The A_- EOM gives a constraint $\partial_-(\partial \cdot A) = 0$, $i.e.$ $k_-(-k_-A_+ + k_iA^i) = 0$. Thus if $k_- \neq 0$, then $-k_-A_+ + k_iA^i = 0$. Now solve for A_+ in terms of A_i , $i.e.$ $A_+ = \frac{1}{k_-}(k_iA_i)$.

Note: $\partial_-(\partial \cdot A) = 0$ means $\partial \cdot A = F(X^+, x^i)$. Residual X^--independent gauge transformations $A'_\mu = A_\mu + \partial_\mu \lambda,\ \mu \neq X^-$, can be used to fix $\partial \cdot A = 0$, for $k_- \neq 0$.
Thus for $k_- \neq 0$ modes, we can fix gauge completely $\Rightarrow A_+, A_i$ are gauge-invariant.
In a general gauge: $\tilde{A}_\mu = e^{-\Phi/2}(A_\mu + \partial_\mu \chi)$, where $\chi = -\partial_-^{-1} A_-$ is uniquely defined if $k_- \neq 0$.

$Caveat!$ $k_- = 0$ subtleties of lightcone gauge.

Tilde variables cont'd.

Curved metric $\tilde{g}_{\mu\nu} = e^f \eta_{\mu\nu}$: Dilaton couples to dimension $\Delta = 4$ operators, so no dressing factors since dressed source is $Je^{\frac{4-\Delta}{2}f}$.

Interaction terms are of the form $e^{k\Phi(X^+)}\mathcal{O}(x)$. From earlier arguments, $e^{\Phi(X^+)}$ is a lightlike source $\Rightarrow$ no particle production. Dilaton couples with positive powers $\Rightarrow$ all interactions die out when the dilaton vanishes. Thus as $X^+ \rightarrow 0$, the $\tilde{A}$ theory is becoming free.

Note: in the regulated theory, $e^f = (|\tanh X^+| + \epsilon)^2$, there is no singular behaviour at all. Only possible singular behaviour for $\epsilon \rightarrow 0$ arises near $X^+ \rightarrow 0$. But here, all interactions die.

Thus the gauge theory is nonsingular.

Other observables, bulk 2pt fn

* Operator relation: $\mathcal{O} \equiv e^{-\Phi}\,\mathrm{Tr}\,F^2 = \mathrm{Tr}\,\tilde{F}^2 - \frac{1}{2}\partial_-[(\partial_+\Phi)\tilde{A}_i\tilde{A}^i]$.
Then $\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \langle\mathrm{Tr}\,\tilde{F}^2(x)\,\mathrm{Tr}\,\tilde{F}^2(y)\rangle + divergent$,
since $\Phi \sim \sqrt{8}\log X^+ \ \Rightarrow\ \partial_+\Phi \sim \frac{1}{X^+}$ near $X^+ = 0$, so that
$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle$ diverges. However, in our example, $\langle\mathrm{Tr}\,F^2(x)\,\mathrm{Tr}\,F^2(y)\rangle$
etc vanish as $X^+ \to 0$ since $e^\Phi \to 0$ faster than $\partial_+\Phi$ diverges.

* For operator $\mathcal{O}$ dual to bulk scalar ($e.g.$ dilaton), bulk 2-pt function is
$\langle e^{\frac{f(x)\Delta}{2}}O(x)e^{\frac{f(x')\Delta}{2}}O(x')\rangle = e^{\frac{f(x)(\Delta-1)}{2}}e^{\frac{f(x')(\Delta-1)}{2}}(\frac{\Delta\lambda}{\Delta X^+})^{1-\Delta}\,\frac{1}{[(\Delta\vec{x})^2]^\Delta}$.
When $x \sim x'$, then $\frac{\Delta\lambda}{\Delta X^+} \sim \frac{d\lambda}{dX^+} = e^f$ giving $\frac{1}{[(\Delta\vec{x})^2]^\Delta}$.

However in singular backgrounds, for both $X^+, X'^+ \to 0$, this
depends on how the limit is taken: in our example, $\lambda \sim (X^+)^3$, giving
$(\frac{X^+}{(X')^+})^{\Delta-1}[(\frac{X^+}{(X')^+})^2 + \frac{X^+}{(X')^+} + 1]^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}$.

More on dual variables

This disagreement between bulk and gauge theory expectations is not surprising: bulk calculation fails near the singularity. Also these bulk modes, $e.g.$ dilaton, couple to operators made out of A_μ variables.

Good gauge theory variables are the $\tilde{A}_\mu$: bulk duals ? Hard to identify clearly. Operators such as $\tilde{F}_{ij} = e^{-\Phi/2} F_{ij}$ are local.
However operators $e.g.$ $\tilde{A}_\mu$, or $\tilde{F}_{+\nu} = e^{-\Phi/2} F_{+\nu} - \frac{1}{2} e^{-\Phi/2} (A_\nu \partial_+ \Phi)$, are not local in terms of $F_{\mu\nu}$, since A_μ cannot be expressed locally in terms of $F_{\mu\nu}$. A complete set of gauge invariant operators must include these. Thus the $\tilde{A}_\mu$ are possibly nonlocal, $i.e.$ their duals (good bulk variables) are stringy (recall $e.g.$ Wilson loop).
Also note: in usual AdS/CFT, $\alpha' \sim \frac{1}{g_{YM}^2 N} = \frac{1}{e^\Phi N}$.
This further suggests that α' (stringy) effects become important near the singularity (crude bulk worldsheet analysis corroborates this).

Open questions

- Work in progress: more detailed understanding of $\mathcal{N}=4$ SYM with this time-dependent coupling, loop amplitudes, etc. Spacelike singularities ?
- What is the bulk resolution of these null singularities ? Stringy physics near these singularities ?

  D-brane dynamics in time-dependent backgrounds ?
- More severe (spacelike) cosmological singularities ? Realistic cosmologies ?