Quantum Mechanics II:
Endsemester examination (take-home)

Total: 50 marks

Time: 2 days.

(1) Consider the outermost electron in a hydrogen-like atom of atomic number Z. Let us approximate the effective screened potential due to the nucleus and the $(Z-1)$ inner electrons by $V = -e^2(\frac{1+(Z-1)e^{-(Z-1)r/a_0}}{r})$, *i.e.* $V \sim -\frac{Ze^2}{r}$ for $r \lesssim \frac{a_0}{Z-1}$ while $V \sim -\frac{e^2}{r}$ for $r \gg \frac{a_0}{Z-1}$.

Consider a (normalized) trial groundstate wavefunction for the outermost electron in terms of a basis $\psi_1 = e^{-r/a_0}$ and $\psi_2 = e^{-Zr/a_0}$, the wavefunctions for charge e and charge Ze, the two coefficients in the variational state being the variational parameters. Find the expectation value of the ground state energy. Extremize with respect to the parameters to find the "best fit" wavefunction for the outermost electron. [9 mks]

(2) (a) Consider a particle of mass m and energy E moving in a 1-dimensional potential $V = V_0(1-\frac{x^2}{a^2})$. Find the turning points and use the quasi-classical (WKB) approximation to calculate the transmission coefficient through the barrier. [5 mks.]

(b) Consider a potential $V = \frac{V_0}{1-\frac{|x|}{a}}$ with $|x| \leq a$. Find the quasi-classical (WKB) quantization condition for a bound state with energy E, the particle mass being m.

[4 mks]

(3) Consider a 2-state system with energy eigenvalues E_1, E_2, with $E_1 < E_2$. Consider a potential that switches on at $t = 0$ and is of the form $V_{12} = V_0$ for $0 < t < t_0$, and then slowly turns off as $V_{12} = V_0 e^{-(t-t_0)/\tau}$ for $t > t_0$ (assume $V_{11}, V_{22} = 0$, all parameters in V being real). The system is initially in the lower level $|1\rangle$.

(a) Find the occupancy coefficients $c_1(t), c_2(t)$ of the states $|1\rangle, |2\rangle$ as a function of time by solving the problem exactly. For the differential equation you will find for $t > t_0$, use a new coordinate $T \sim e^{-t/\tau}$ to solve the equation in terms of Bessel functions. Using this solution (and the initial conditions) or directly from the differential equation, find the asymptotic form at late times of $c_1(t), c_2(t)$. [10 mks]

(b) Find the coefficients $c_1(t), c_2(t)$ by using first order time-dependent perturbation theory and compare with the exact result in (a). [7 mks]

(4) (a) Consider a Hydrogen atom in a state $(n, l, j, m_j) = (n, n-1, n-\frac{3}{2}, n-\frac{3}{2})$. Calculate the fine structure corrections to the energy (incorporating the relativistic, Darwin and spin-orbit contributions). [4 mks]

(b) Consider the system now in the presence of a magnetic field $\vec{B} = \hat{z}B$. In the weak-field (Zeeman) limit, calculate the further corrections to the energy due to the magnetic field coupling. [4 mks]

(5) Consider a potential of the form $V(r) = -\frac{\alpha}{r}$ for $R \leq r \leq 2R$, with $V = 0$ for $0 < r < R, r > 2R$. Find the first order Born scattering amplitude for a particle of mass m and momentum p. Find also the differential cross-section in the low energy limit $p \to 0$. [7 mks]