General Relativity:
Endsemester (take-home) examination

Total: 50 marks

For some questions below, you may find it useful to use the Maple GR package as a check of your calculations of various quantities.

(1) Consider a spacetime of the form $ds^2 = e^{f(u)}(-2dudv + dx^2 + dy^2)$, written in terms of lightcone coordinates u, v. One of the coordinates, say u, can be taken to be the lightcone time coordinate. Note that flat (Minkowski) spacetime in lightcone coordinates corresponds to $f(u) = 0$.
(a) Define a suitable orthonormal basis of 1-forms e^a so that the metric takes the form $ds^2 = -2e^+e^- + (e^1)^2 + (e^2)^2$. Find the corresponding (spin) connection 1-forms (in the obvious diagonal frame, you should find three independent nonzero 1-forms). [7 mks]
(b) Find the curvature 2-forms from the spin connection above and thereby obtain the coordinate basis expressions for the Ricci tensor components. As a check, show that these match with the Ricci components obtained by a direct calculation using the Christoffel symbols. [*Hint: the Ricci tensor is rather simple!*] [7 mks]
(c) Simplify Einstein's equations for this system with a stress tensor arising from a matter distribution sourced by a massless scalar field with no potential energy and show that the resulting equations take the form $R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi$. For the simple case where the scalar field depends only on u, i.e. $\phi = \phi(u)$, simplify the stress tensor and the Einstein equation. You should find a single simple equation relating $f(u)$ and $\phi(u)$ for the resulting family of cosmological solutions. [6 mks]

(2) We have seen that FRW models are rather successful cosmological models for our Universe based on homogeneity and isotropy. Physicists have also studied cosmological models dropping these assumptions. One of the simplest such anisotropic models is called the Kasner cosmology. Consider a spacetime of the form

$$ds^2 = -dt^2 + \sum_{i=1}^{3} e^{h_i(t)} dx^i dx^i \ .$$

(a) Work out the Ricci tensor components either in the coordinate basis or in an orthonormal frame. [6 mks]
(b) We want to look for Ricci-flat solutions (i.e. zero Ricci tensor). Show that $e^{h_i(t)} = t^{2p_i}$

are possible solutions to these equations, where p_1, p_2, p_3 are parameters. Show also that these parameters satisfy the conditions

$$p_1 + p_2 + p_3 = 1 \ , \qquad p_1^2 + p_2^2 + p_3^2 = 1 \ .$$

It is worth mentioning here that there are various more general families of such cosmologies, based on the Bianchi classification of homogenous spatial 3-geometries. [6 mks]

(3) We know that a Reissner-Nordstrom black hole of mass m and charge Q has $g_{tt} = -(1 - \frac{2m}{r} + \frac{Q^2}{r^2})$.

(a) Convert this expression to SI units by sticking back the fundamental constants G, c, ϵ_0 etc so that the resulting terms are dimensionally correct. [2 mks]

(b) By studying the known values for the mass and charge of an electron, figure out if an electron (treated as a classical energy source) can be considered as a black hole (i.e. without a naked singularity). [4 mks]

(c) Compare the the Schwarzschild radius of a black hole of mass m_e with the "classical electron radius" r_e defined by $\frac{e^2}{4\pi\epsilon_0 r_e} = m_e c^2$. [2 mks]

(4) Show that a near-extremal charged black hole, i.e. with $m \sim Q$, has positive specific heat. [5 mks]

(5) We have seen that the Friedmann equation can be written in the form $H^2 a^2 (1 - \Omega(t)) = -k$, where $H(t)$ is the Hubble parameter and $\Omega(t)$ is the density parameter at that epoch. Suppose the universe underwent a short inflationary phase where the scale factor evolved exponentially with constant Hubble parameter H. You can get an estimate for H by using the Friedmann equation in the form $H^2 \sim G\rho$ and assuming near-Planckian energy densities ρ at the beginning of inflation. Also assume that $1 - \Omega \sim \mathcal{O}(1)$ at the beginning of inflation. Obtain an order-of-magnitude estimate of how long such an inflationary phase needs to last so that the density parameter is driven to unity to the required degree of apparent fine-tuning (about 10^{-60}) for standard Big-Bang cosmology to be a reasonable description subsequently. [5 mks]