Classical Mechanics II:
End-sem examination

Total: 50 marks

(1) Consider a planar pendulum, i.e. a mass point m hanging from a fixed point on the ceiling by a string (length l), swinging in a vertical plane.
(a) Treating the pendulum as a rigid body, draw a figure showing Euler angles and find appropriate inertia tensor components I_1, I_2, I_3, for this symmetric top motion about the ceiling point of suspension. Choose some convenient body (moving) x_1, x_2, x_3-axes, and explain clearly your choice. [4 mks.]
(b) Write out the angular velocity components with respect to these moving axes and write Euler's equations for the rotational dynamics of this pendulum. Discuss how your answer can equivalently be obtained from any other way, e.g. using a Lagrangian. As a check of your analysis, see if you can obtain the expected frequency of oscillations for small swing angle. [5 mks.]

(2) (a) In the above problem of the planar pendulum, work out the Hamiltonian H from the definition, starting with say the Lagrangian. [4 mks].
(b) What is the angular momentum vector $\vec{L}$ for this planar motion ? Evaluate the Poisson bracket $\{H, \vec{L}\}$. [4 mks]
(c) Write Hamilton's equations for this system. Show that the Poisson bracket and Hamilton's equations are in accordance with the evolution of the angular momentum as obtained using the Lagrangian. [4 mks]

(3) For the 1-dim discrete map $x_{n+1} = (e^{x_n} - 1)^2$, identify the fixed points and discuss their stability. [5 mks]

(4) Consider the transformation $Q = pg(q), P = f(q)$, from variables (q, p) to (Q, P), where $f(q), g(q)$ are two arbitrary functions of q.
(a) What relation(s) must the functions $f(q), g(q)$ satisfy for this transformation to be canonical ? [4 mks]
(b) In this case, find a suitable generating function as a function of q, Q. [4 mks]

(5) A particle moves subject to a time-dependent Lagrangian $L_x = \frac{1}{2}Ct^A\dot{x}^2$, with C, A constant parameters. The redefinition to the variable $y = t^{A/2}x$ transforms this to the form $L_y = \frac{C}{2}(\dot{y}^2 - \frac{A(2-A)}{4t^2}y^2)$, with $L_x = L_y - \frac{d}{dt}(\frac{CAy^2}{2t})$.

(a) Find the conjugate momenta p_x, p_y, and the Hamiltonians H_x, H_y in the x- and y-variables. [4 mks]

(b) From the difference $H_y - H_x$, guess a generating function $F(x, p_y)$ for the canonical transformation between the x, y-variables. [5 mks]

(c) Verify the relations between x, y, p_x, p_y from the differential relations in terms of the generating function $F(x, p_y)$. As a basic check, note that for $A = 0$, these should reduce to the identity transformation. [7 mks]