Defining Gromov Witten invariants

Barbara Fantechi

Chennai Mathematical Institute
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Outline of this lecture

Quotient stacks

From complexes to stacks

Notation We fix a base scheme S, a scheme X over S, and a group scheme G over S with a (left) action on X.

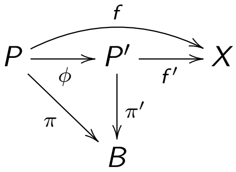
Definition

The *quotient stack* $[X/G]$ is the pseudofunctor $(\text{sch}/S) \to (\text{grpd})$ defined as follows. For B an S-scheme

1. the objects of $[X/G](B)$ are triples (P, π, f) such that

$$\pi : P \to B \text{ is a } G\text{-torsor and } f : P \to X \text{ is } G\text{-equivariant.}$$

2. morphisms $(P, \pi, f) \to (P'\pi', f')$ are G-equivariant morphisms $\phi : P \to P'$ such that the following diagram commutes:



Exercise

1. Show that all morphisms in $[X/G](B)$ are isomorphisms.
2. For $g : B' \to B$, define $g^*(P, \pi, f) = (P', \pi', f')$ where $P' := P \times_B B'$ and π' and f' are defined by the commutative diagram

$$\begin{array}{ccccc} & & \overset{f'}{\frown} & & \\ P' & \xrightarrow{p_P} & P & \xrightarrow{f} & X \\ {\scriptstyle \pi' = p_{B'}} \downarrow & & \downarrow {\scriptstyle \pi} & & \\ B' & \xrightarrow{g} & B & & \end{array}$$

3. Show that this defines a pseudofunctor, which is a stack in the Zariski topology.
4. If you are familiar with descent theory, show that it is a stack in the étale topology.

Define a (tautological) morphism $\tau : X \to [X/G]$ by the element $(G \times X, p_X, a) \in [X/G](X)$ where G acts on $G \times X$ via $g_1(g_2, x) = (g_1 g_2, x)$ and a is the action.

Theorem

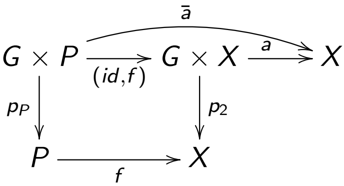
Let B be an S-scheme, and $b = (P, \pi, f) \in [X/G](B)$. There is a natural 2-cartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tau} \\ B & \xrightarrow[b]{} & [X/G] \end{array}$$

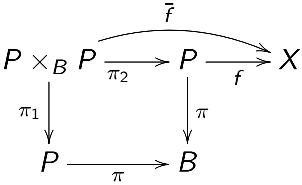
Proof.

We first define an isomorphism $\alpha : \tau \circ f \Rightarrow b \circ \pi$ to make the diagram 2-commutative; we then prove that the induced morphism $P \to B \times_{[X/G]} X$ is an equivalence. □

The morphism $\tau \circ f$ is defined by the diagram with cartesian square



where G acts on $G \times P$ via $g_1(g_2, p) = (g_1 g_2, p)$ and $p_P(g, p) = p$.
The morphism $b \circ \pi$ is defined by the diagram with cartesian square



where G acts on $P \times_B P$ via $g(p_1, p_2) = (p_1, gp_2)$. We want to define $\alpha : G \times P \to P \times_B P$ which is G-equivariant and satisfies $p_P = \pi_1 \circ \alpha$ and $\bar{a} = \bar{f} \circ \alpha$.
We define $\alpha(g, p) = (p, gp)$ and verify it has the required properties.

The 2-commutative diagram

$$\begin{CD} P @>{f}>> X \\ @V{\pi}VV @VV{\tau}V \\ B @>>{b}> [X/G] \end{CD}$$

induces a morphism $P \to B \times_{[X/G]} X$. To prove that it is an equivalence, we construct an explicit inverse.
Let Y be a scheme. A morphism from Y to the fiber product corresponds to a 2 commutative diagram

$$\begin{CD} Y @>>> X \\ @VVV @VV{\tau}V \\ B @>>{b}> [X/G] \end{CD}$$

which means an isomorphism $\beta : G \times Y \to P \times_B Y$. We obtain from this a morphism $Y \to P$ by

$$Y \xrightarrow{(e,id)} G \times Y \xrightarrow{\beta} P \times_B Y \xrightarrow{\pi_P} P.$$

As a corollary to this theorem, we have proved that if $G \to S$ is étale or smooth, then so is $X \to [X/G]$ thus providing an atlas.

Exercise Show that for a scheme B, any morphism $B \to [X/G]$ is strongly representable. Indeed, if $C \to [X/G]$ is another morphism from a scheme, the fiber product $B \times_{[X/G]} C$ is quasi projective over $B \times C$.

When is the quotient stack $[X/G]$ algebraic?

The answer is complicated, since it depends on the definition of algebraic stack, which in turn isn't always the same.

A good rule of thumb is that you want $G \to S$ to be smooth, or at least flat; this way $\tau : X \to [X/G]$ gives an atlas.

A sufficient condition (i.e., one that works with all definitions I know) is that X be of finite type and G be a closed subgroup of $GL(N)$ or $\mathbb{P}GL(N)$ for some N.

Example: $\overline{M}_g$ as quotient stack

Fix $g \geq 2$, and $m \geq 3$. Let $P(t) = m(2g-2)t + 1 - g$ and $N := 1 - g + m(2g-2) = P(1)$.
Let C be a prestable genus g curve. Then $H^1(C, \omega_C^{\otimes m}) = 0$ and C is stable if and only if $\omega_C^{\otimes m}$ is very ample.
Let $U \subset Hilb^P(\mathbb{P}^N)$ be the open subscheme parametrising stable curves, and $V \subset U$ the closed subscheme parametrizing curves $C \subset \mathbb{P}^N$ such that $\mathcal{O}_C(1)$ is isomorphic to $\omega_C^{\otimes m}$.
Exercise Show that $\overline{M}_g$ is isomorphic to the stack quotient $[V/G]$ where $G = Aut(\mathbb{P}^N)$.
Advantages: we can think, e.g., of line bundles on $\overline{M}_g$ as G-equivariant line bundles on V.
Disadvantages: it's non-canonical (we *choose* an m) and *unnatural*. For instance, proving that $\overline{M}_g$ is smooth is way easier than proving that V is smooth.

Let X be an algebraic stack (if this helps, you can assume it is a scheme; the stackiness of X will play no role in what follows). We consider the following 2-category $M(X)$.

1. the objects of $M(X)$ are morphisms of coherent sheaves $d_E : \mathcal{E}_{-1} \to \mathcal{E}_0$ on X with $\mathcal{E}_0$ locally free.
2. morphisms $\phi = (\phi_{-1}, \phi_0)$ are commutative diagrams

$$\begin{array}{ccc} \mathcal{E}_{-1} & \xrightarrow{d_E} & \mathcal{E}_0 \\ \downarrow{\scriptstyle \phi_{-1}} & & \downarrow{\scriptstyle \phi_0} \\ \mathcal{F}_{-1} & \xrightarrow[d_F]{} & \mathcal{F}_0 \end{array}$$

3. a 2 morphism $\alpha : \phi \Rightarrow \psi$ is a homomorphism $\alpha : \mathcal{E}_0 \to \mathcal{F}_{-1}$ such that

$$\psi_0 - \phi_0 = d_F \circ \alpha \quad \text{and} \quad \psi_{-1} - \phi_{-1} = \alpha \circ d_E.$$

Let us now define a contravariant 2-functor H from $M(X)$ to algebraic stacks.

- $H(d_E : \mathcal{E}_1 \to \mathcal{E}_0) := [E_1/E_0]$ where $E_i = Spec\, Sym\, \mathcal{E}_{-i}$ is an abelian group scheme over X, and the homomorphism of group schemes $E_0 \to E_1$ induced by d_E makes E_0 act on E_1 by translations.
- Let $\phi : \mathcal{E}_\bullet \to \mathcal{F}_\bullet$. We define $H(\phi)$ as follows. Let $g : B \to X$ be a morphism with B a scheme, and $(P, \pi, f) \in [F_1/F_0](B)$. That is, P is a g^*F_0 torsor and $f : P \to F_1$ is an equivariant morphism.
  The morphism ϕ induces a commutative diagram of abelian group schemes over X

$$\begin{array}{ccc} F_0 & \xrightarrow{\bar{d}_f} & F_1 \\ \Big\downarrow{\scriptstyle \bar{\phi}_0} & & \Big\downarrow{\scriptstyle \bar{\phi}_1} \\ E_0 & \xrightarrow[\bar{d}_E]{} & E_1 . \end{array}$$

We define $H(\phi)(P, \pi, f)$ to be (P', π', f') where

- $P' = P \times_X E_0 / F_0$ By this we mean that $P \times_X E_0 \to P'$ is an F_0-torsor, where F_0 acts via $f_0(p, e_0) = (f_0 \cdot p, e_0 - \bar{\phi}_0(e_0))$.
- **Exercise** Prove that $P \times_X E_0 \to B$ induces $\pi' : P' \to B$ making P' into an E_0-torsor.
- Finally, define $f' : P' \to F_1$ as the morphism induced by

$$P \times_X E_0 \to F_1 \text{ given by } (p, e_0) \mapsto \phi_1(f(p)) + \bar{d}_E(e_0).$$

Exercise Associate to each $\alpha : \phi \to \psi$ a 2-morphism $H(\alpha) : H(\psi) \to H(\phi)$.

Let $\phi : \mathcal{E}_\bullet \to \mathcal{F}_\bullet$ be a morphism. We call $\ker\phi$ the induced morphism $\ker d_E \to \ker d_F$, and similarly for $coker\phi$.
If you prefer you can call the first $h^{-1}(\phi)$ and the second $h^0(\phi)$.

Theorem

1. *The morphism $H(\phi)$ is representable iff $coker\phi$ is surjective;*
2. *$H(\phi)$ is a closed embedding iff $coker\phi$ is an isomorphism and $\ker\phi$ is surjective;*
3. *$H(\phi)$ is an equivalence iff $coker\phi$ and $\ker\phi$ are isomorphisms.*

Key idea in the proof This is a local statement in X: not just Zariski local, but étale and smooth local.
So we can assume $X = Spec\, A$ where A is a f.g. $\mathbb{C}$-algebra, and that $\mathcal{E}_0 = \mathcal{O}_X^{\oplus e_0}$, $\mathcal{F}_0 = \mathcal{O}_X^{\oplus f_0}$. We can also always pass to a smaller open affine.

1. The fibres of $H(\phi)$ are rigid groupoids iff $coker\phi$ is surjective (category theory).
2. Let $\mathcal{G}$ be any locally free sheaf, $\mathcal{E}'_i := \mathcal{E}_i \oplus \mathcal{G}$, $d'_e := (d_e, id)$. Then the natural morphisms
$$[E_1/E_0] \to [E'_1/E'_0] \to [E_1/E_0]$$
are equivalences.
3. If $coker\phi$ is surjective, working locally and using 2. we can assume $\mathcal{E}_0 = \mathcal{F}_0 \Rightarrow E_0 = F_0$. Moreover $\ker(\phi)$ is surjective (resp. an isomorphism) iff ϕ_{-1} is.
4. If $E_0 = F_0$, then the diagram
$$\begin{array}{ccc} F_1 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ [F_1/F_0] & \longrightarrow & [E_1/E_0] \end{array}$$
is cartesian with smooth surjective vertical arrows, hence $H(\phi)$ is a closed embedding (resp. an isomorphism) iff $F_1 \to E_1$ is.

Let X be a scheme or an algebraic stack. A sheaf of graded quasicoherent algebras $\mathcal{A} = \oplus_{n \geq 0} \mathcal{A}_n$ on X satisfies (†) is $\mathcal{O}_X \to \mathcal{A}_0$ is an isomorphism, $\mathcal{A}_1$ is coherent and generates $\mathcal{A}$ as a sheaf of algebras.

Definition

A *cone* over X is an affine morphism $p : C \to X$ with a $\mathbb{G}_m$ action on C such that p is $\mathbb{G}_m$ invariant and the induced grading on $p_* \mathcal{O}_C$ satisfies (†). A morphism of cones is a $\mathbb{G}_m$-equivariant morphism of schemes over X. A cone C is *abelian* if the natural morphism $Sym^* \mathcal{A}_1 \to \mathcal{A}$ is an isomorphism, where $\mathcal{A} = p_* \mathcal{O}_C$.

Lemma

The functor $\mathcal{F} \mapsto Spec\, Sym^* \mathcal{F}$ induces a natural equivalence of categories

$$Coh(X)^{op} \to (abelian\ cones).$$

Remark We can identify $\mathrm{Coh}(X)$ with $D^0_{coh}(X)$.

Definition

A *[abelian] stack cone* over X is a morphism of algebraic stacks $p : C \to X$ with a $\mathbb{G}_m$ action on C such that

1. p is $\mathbb{G}_m$ equivariant;
2. p is locally isomorphic to a quotient $[C/E]$ where C is a [abelian] cone and E is a vector bundle acting equivariantly on C.

A [2-]morphism of cone stacks is a $\mathbb{G}_m$ equivariant [2-]morphism. An abelian cone stack is a *vector bundle stack* of rank $r \in \mathbb{Z}$ if it is locally isomorphic to a quotient $[E_1/E_0]$ with E_i vb of rank r_i and $r = r_1 - r_0$.

Warning I haven't defined what is a group action on a stack.

Theorem

The 2-functor H induces an equivalence of categories between $D_{coh}^{-1,0}(X)^{op}$ and the homotopy category of abelian cone stacks. Moreover, $H(\mathcal{E})$ is a vector bundle stack iff $\mathcal{E}$ is locally isomorphic to a complex of locally frees.

The proof is inspired by a similar theorem of Deligne.
I'm cheating! We shouldn't use $\mathbb{G}_m$-actions but (multiplicative) $\mathbb{A}^1$-actions.
Key idea: If V and W are vector spaces over $\mathbb{C}$, then they have natural structures of algebraic varieties and a morphism $\phi : V \to W$ is linear iff it is $\mathbb{G}_m$ equivariant.
The theorem follows immediately from the properties of H if X has enough locally frees.

Definition

Let $f : X \to Y$ be a morphism of DM type of algebraic stacks. We write N_f for the abelian cone stack associated to $\tilde{L}_f = \tau_{\geq -1} L_f$,

If f factors as $p \circ i$ with p smooth and DM type and i a closed embedding, then $N_f = [N_i / i^* T_p]$. It contains $[C_i / i^* T_p]$ as a closed substack.

Lemma

There is a unique closed substack C_f of N_f such that it locally induces $[C_i / i^* T_p]$.

We call C_f the normal cone to f. If Y is irreducible of dimension $d \in \mathbb{Z}$, then C_f is pure dimensional of dimension d.
There is a one-parameter degeneration of $f : X \to Y$ to $X \to C_f$ (the vertex of the cone).

Theorem

Let $p : E \to X$ *be a vector bundle stack of rank* r*. Then* $p^* : A_d(X) \to A_d + r(X)$ *is an isomorphism for all* r*.*

Definition

Let $\phi : \mathcal{E} \to \tilde{L}_f$ be a morphism in $D_{coh}^{-1,0}(X)$.
We say it is an *obstruction theory* if the induced morphism $N_f \to E = H(\mathcal{E})$ is a closed embedding.
It is a *perfect obstruction theory of rank* r if E is a vector bundle stack of rank $-r$.

Corollary

To a perfect obstruction theory we can associate a virtual pullback

$$A_d(Y') \to A_{d+r}(X')$$

for every base change $f' : X' \to Y'$ *of* $f : X \to Y$*.*

Lemma

Let $\phi : \mathcal{E} \to \tilde{L}_f$ in $D^b_{coh}(X)$. Then ϕ is an obstruction theory iff for every $x \in X$

- $h^0(x^*\tilde{L}_f^\vee) \to h^0(x^*\mathcal{E}^\vee)$ *is an isomorphism;*
- $h^1(x^*\tilde{L}_f^\vee) \to h^1(x^*\mathcal{E}^\vee)$ *is an injective.*

In other words, being an obstruction theory is equivalent to inducing at every point a relative tangent and obstruction space.
Example The forgetful morphism $F : \overline{M}_{g,n}(V, \beta) \to \mathfrak{M}_{g,n}$ has a perfect obstruction theory $\mathcal{E} \to \tilde{L}_F$ where $\mathcal{E} = (R\pi_* f^* T_V)^\vee$.