Defining Gromov Witten invariants

Barbara Fantechi

Chennai Mathematical Institute
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Outline of this lecture

Curve classes and expected dimensions

Normal cone of a closed embedding

Fulton-McPherson intersection theory

Stacks make dreams come true

Quotient stacks

Curve classes and expected dimension -1

Let V be a smooth projective variety. A *curve class on X* is a group homomorphism $\beta : Pic(X) \to \mathbb{Z}$ such that $Pic^0(X) \subset \ker \beta$. If (C, x_i, f) is a (pre)stable map to V, we define its class by

$$\beta(L) := \deg_C f^* L.$$

If (C, x_i, π, f) is a family of prestable maps over a scheme S and β is a curve class, the set of points $s \in S$ such that $(C_s, x_{i,s}, f_s)$ has class β is open in S. If $S_\beta = S$, we say it is a family of class β. We define the algebraic moduli stack $\overline{M}_{g,n}(V, \beta)$ as the pseudofunctor mapping a scheme S to the groupoid of families of stable maps over S of class β.

Curve classes and expected dimension -1

For any embedding $V \subset \mathbb{P}^N$ and any curve class β on V, let $d := \beta(\mathcal{O}_V(1))$. Then $\overline{M}_{g,n}(V,\beta)$ is an open and closed substack of $\overline{M}_{g,n}(V,d)$. In particular, it is proper.
Last lecture we proved that the forgetful morphism

$$F : \overline{M}_{g,n}(V,\beta) \to \mathfrak{M}_{g,n}$$

has relative expected dimension $(1-g)\dim V + \beta(\det T_V)$.
An alternative approach is to use classical topology and define $\beta := f_*[C] \in H_2(V,\mathbb{Z})$; then the relative dimension is $(1-g)\dim V + \beta \cdot c_1(T_V)$. The advantage of the former is that it works in any characteristic.

Normal cone

Let $i : X \to Y$ be a closed embedding of schemes, $\mathcal{I} = \mathcal{I}_{X/Y}$ the ideal sheaf. The *normal cone of X in Y* is

$$C_{X/Y} = Spec \oplus i^*\mathcal{I}^n.$$

Theorem

If Y is irreducible and reduced of dimension n, then $C_{X/Y}$ is pure dimensional of dimension n.

Proof.

Let Y' be the blow-up of $Y \times \mathbb{A}^1$ along $X \times 0$. Y' is reduced, irreducible of dimension $n+1$. Let E be the exceptional divisor of the blow-up; by definition, it is pure-dimensional of dimension n. E contains an open subscheme isomorphic to $C_{X/Y}$. □

Normal sheaf

Let $i : X \to Y$ be a closed embedding. Let
$N_{X/Y} := Spec\, Sym^* i^* \mathcal{I}_{X/Y}$.
There is a natural closed embedding $C_{X/Y} \to N_{X/Y}$, and the latter is an abelian group scheme over X.
For every cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y \end{array}$$

there is a natural closed embedding
$C_{X'/Y'} \to g^*(C_{X/Y}) := X' \times_X C_{X/Y}$ and similarly for $N_{X/Y}$.
Let $\mathcal{E} \to i^* \mathcal{I}_{X/Y}$ be a surjection of coherent shaves on X.
It induces closed embeddings

$$C_{X/Y} \to N_{X/Y} \to Spec\, Sym^* \mathcal{E}.$$

Examples of normal cones and sheaves

- Show that there is a natural $\mathbb{G}_m$-action on $C_{X/Y}$ and $N_{X/Y}$ such that $C_{X/Y} \to N_{X/Y}$ is equivariant.
- Show that the natural projection $C_{X/Y} \to X$ is $\mathbb{G}_m$ invariant, and it admits a natural $\mathbb{G}_m$ invariant (zero) section.
- Let Y be a smooth variety and $X \subset Y$ a smooth subvariety, Then $C_{X/Y} = N_{X/Y}$ is the normal bundle of X in Y.
- The same is true if Y is an arbitrary scheme and X is a regularly embedded closed subscheme.
- Compute $C_{X/Y}$ and $N_{X/Y}$ for $Y = \mathbb{A}^2$ and $X = Spec\, \mathbb{C}[x, y]/(x^2, xy, y^2]$. In particular show that $C_{X/Y}$ has a unique irreducible component, isomorphic to a quadric cone.
- Compute $C_{X/Y}$ and $N_{X/Y}$ for $Y = \mathbb{A}^3$ and $X = Spec\, \mathbb{C}[x, y, z]/xy, xz$.
  Show that $C_{X/Y} = N_{X/Y}$, and find its irreducible components (hint: there are two of them).

Chow groups and their functoriality

Let X be a scheme of finite type over $\mathbb{C}$. We define $Z_d(X)$ to be the free abelian group generated by d-dimensional subvarieties of X. We let $Z_*(X) := \oplus_d Z_d(X)$.
We define $Rat_d(X)$ to be the subgroup of $Z_d(X)$ generated by $div_W(r)$, where W is a $(d+1)$-dimensional subvariety of X and r is a nonzero rational function on W.
The Chow group $A_d(X)$ is the quotient of $Z_d(X)$ by $Rat_d(X)$.
Proper pushforward If $f : X \to Y$ is proper, it induces $f_* : Z_d(X) \to Z_d(Y)$ and $f_* : A_d(X) \to A_d(Y)$.
Flat pullback If $f : X \to Y$ is flat of relative dimension r, it induces $f^* : Z_d(Y) \to Z_{d+r}(X)$ and $f^* : A_d(Y) \to A_{d+r}(X)$.
Vector bundle If $\pi : E \to X$ is a rank r vector bundle, then $\pi^* : A_d(X) \to A_{d+r}(E)$ is an isomorphism. We denote $0^!$ its inverse.

Gysin pullback

Let $i : X \to Y$ be a closed embedding, $\phi : \mathcal{E} \to i^*\mathcal{I}_{X/Y}$ a surjection with $\mathcal{E}$ a locally free sheaf of rank r on X.

Theorem

For every cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ j\ } & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[\ i\ ]{} & Y \end{array}$$

we have an induced homomorphism $i^!_{\mathcal{E}} : A_d(Y') \to A_{d-r}(Y)$*, compatible with proper push forward, flat pullbacks and Chern classes.*

If i is a regular embedding of codimension r and ϕ is an isomorphism, we just write $i^!$ and call it *Gysin pullback*.

Definition of Gysin pullback

We review the definition of $i^!_{\mathcal{E}}$. Let $V \subset Y'$ be a d-dimensional variety, $W := V \cap X' := V \times_Y X$. Let $p := g|_W : W \to X$. We have a closed embedding $C_{W/V} \to p^* C_{X/Y} \to p^* E$, where $E = Spec\, Sym\, \mathcal{E}$.

Since V is a variety of dimension d, $C_{W/V}$ is a scheme of pure dimension d; hence it has a fundamental cycle

$$[C_{W/V}] \in A_d(C_{W/V})$$

where each irreducible component appears with its natural multiplicity. We define $i^!_{\mathcal{E}}[V]$ to be the image of $[C_{W/V}]$ via the sequence of homomorphisms

$$A_d(C_{W/V}) \longrightarrow A_d(p^* E) \longrightarrow A_{d-r}(W) \longrightarrow A_{d-r}(X')$$

where the first and third maps are pushforwards via closed embeddings, and the second is $0^!$ for the bundle $g^* E \to W$.

Definition of lci pullback

Theorem

Let $f : X \to Y$ be a morphism of schemes which admits a global factorisation $X \xrightarrow{i} M \xrightarrow{p} Y$ with i a regular embedding of codimension r and p smooth of dimension e.

1. *The relative dimension $e - r \in \mathbb{Z}$ of f does not depend on the factorisation chosen.*
2. *For any base change $X' \to Y'$ of f, the group homomorphism*

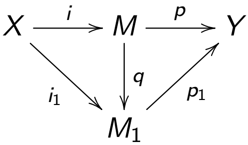
$$f^! := i^! \circ p^* : A_d(Y') \to A_{d+e-r}(X')$$

does not depend on the factorisation chosen.

The homomorphism $f^!$ is called *lci pullback*.

Key ideas in the proof of lci pullback -1

As in last lecture, it is enough to compare factorisations fitting in a diagram



with q a smooth morphism. We saw that the complex

$$\tilde{L}_f := [i^*\mathcal{I}_{X/M} \to i^*\Omega_p]$$

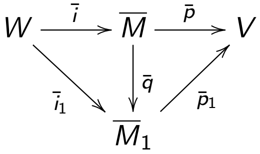
is a well defined element in $D^{-1,0}_{\mathrm{coh}}(X)$ and $e - r$ is its rank.
Let $\mathcal{I} := \mathcal{I}_{X/M}$ and $\mathcal{I}_1 := \mathcal{I}_{X/M_1}$. The locally split exact sequence

$$0 \to i_1^*\mathcal{I}_1 \to i^*\mathcal{I} \to i^*\Omega_q \to 0$$

means that there is an action of i^*T_q on $N_{X/M}$, making it into a principal homogeneous i^*T_q space over N_{X/M_1}.

Key ideas in the proof of lci pullback -2

We now base change the previous diagram via $V \to Y$, yielding



where $\bar{q}$, $\bar{p}$ and $\bar{p}_1$ are smooth, $\bar{i}$ and $\bar{i}_1$ are closed embeddings but not necessarily regular of codimension r.
There is an induced action of $\bar{i}^* T_{\bar{q}}$ on $C_{W/\overline{M}}$, making it into a principal homogeneous $\bar{i}^* T_{\bar{q}}$ space over $C_{W/\overline{M}_1}$. We write this as an *exact sequence of cones and bundles* over W

$$0 \to \bar{i}^* T_{\bar{q}} \to C_{W/\overline{M}} \to C_{W/\overline{M}_1} \to 0.$$

We also have an exact sequence of vector bundles over W

$$0 \to \bar{i}^* T_{\bar{q}} \to \bar{i}^* T_{\bar{p}} \to \bar{i}_1^* T_{\bar{p}_1} \to 0.$$

Dreaming past Fulton-McPherson

Imagine we could combine the two exact sequences above into a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{i}^* T_{\bar{q}} & \longrightarrow & \bar{i}^* T_{\bar{p}} & \longrightarrow & \bar{i}_1^* T_{\bar{p}_1} & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{i}^* T_{\bar{q}} & \longrightarrow & C_{W/\overline{M}} & \longrightarrow & C_{W/\overline{M}_1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C_f & & C_f^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with the two vertical sequences also exact.

Dreaming past Fulton-McPherson -2

It would immediately follow that there is a canonical isomorphism $C_f \to C^1_f$, and that each is a cone of pure dimension dim V.
Parts of this dream are real:

- the natural morphism $\bar{i}^*\mathcal{I}_{W/\overline{M}} \to \bar{i}^*\Omega_{\bar{p}}$ induces a homomorphism of abelian group schemes (over W) $\bar{i}^* T_{\bar{p}} \to N_{W/\overline{M}}$.
- this induces an action of $\bar{i}^* T_{\bar{p}}$ on $C_{W/\overline{M}}$
- the morphism $C_{W/\overline{M}} \to C_{W/\overline{M}_1}$ is equivariant with respect to these actions and the homomorphism $\bar{i}^* T_{\bar{p}} \to \bar{i}^* T_{\bar{p}_1}$.
- However, in general this action is far from being fixed point free. In fact, if $f : X \to Y$ is smooth, and we choose $M = X$, $i = id_X$ and $p = f$, $C_{X/X} = X$, $T + p = T_f$, and the action is the trivial action.

Quotient groupoids -1

As a warm-up, we describe how groupoids allow us to take quotients as if every action was free. Let X be a set, G a group acting on X on the left. Let Y be the quotient set X/G, and $p : X \to Y$ the quotient map.

Lemma

There is a natural commutative diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ \Big\downarrow{\scriptstyle pr_X} & & \Big\downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & Y \end{array}$$

where a is the action. It is cartesian if and only if the action is fixed point free.

Quotient groupoids -2

We define the *quotient groupoid* $[X/G]$ as follows. The objects are the elements of X. For $x, y \in X$, we define

$$Mor(x, y) := \{g \in G \mid g \cdot x = y\}.$$

Exercise

1. Define composition of morphisms, and show that $[X/G]$ is a groupoid.
2. Define a morphism $\pi : X \to [X/G]$ sending each object to itself.
3. Show that there is a (2-)cartesian diagram of groupoids

$$\begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ \downarrow{\scriptstyle pr_X} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\pi]{} & [X/G] \end{array}$$

From quotient groupoids to quotient stacks -1

We now want to address the same problem in the language of stacks.

Notation We fix a scheme S, a group scheme G over S, and a scheme X over S with a G action.

Let $h_X, h_G : (\text{sch}/S) \to (\text{sets})$ be the Yoneda functors (i.e., $h_X(B) = Mor_S(B, X)$).

Giving G a structure of group scheme over S is equivalent to lifting h_G to a functor from schemes to groups, which we also denote h_G.

The action of G on X translates into a natural transformation $h_G \times h_X \to h_X$ such that, for any scheme B over S, the induced map $h_G(B) \times h_X(B) \to h_X(B)$ is an action of the group $h_G(B)$ on the set $h_X(B)$.

From quotient groupoids to quotient stacks -2

It is possible to define a pseudofunctor $[X/G]^w : (\text{sch}) \to (\text{grpd})$ by

$$[X/G]^w(B) = [h_X(B)/h_G(B)],$$

where the right hand side is the quotient groupoid.
However, I added the superscript w to $[X/G]^w$ to mean that this is the *wrong* definition.
We know that, if F is a sheaf of abelian groups on a topological space, and F' a sub sheaf of abelian groups, the quotient sheaf F" cannot be defined as $F"(U) = F(U)/F'(U)$.
The same problem happens in the stack context; the pseudofunctor $[X/G]^w$ defined above isn't a stack, i.e., we cannot defined objects locally and glue them using a cocycle condition.

G torsors

Let S be a scheme, G a group scheme over S, and X an S-scheme with a G-action.

We say that X is a trivial G-torsor over an S-scheme Y if we are given a G-invariant morphism $\pi : X \to Y$ such that there exists a section $s : Y \to X$ with the property that the induced morphism $G \times_S Y \to X$ given by $(g, y) \mapsto g \cdot s(y)$ is an isomorphism. The section s is called a trivialisation of X.

We say that X is a G-torsor over Y if there exists an étale, surjective morphism $Y \to Y$ such that $X' := X \times_Y Y'$ is a trivial G-torsor over Y'.

Exercise Show that the pair (π, Y), if it exists, is unique up to canonical isomorphism.

Exercise Show that $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ is a $\mathbb{G}_m$-torsor.

Show that any étale finite cover of degree 2 is a μ_2 torsor, where μ_2 is the group $\{+1, -1\}$.

G torsors

Let X be a G torsor over Y. Then for every morphism $f : Y' \to Y$, the fiber product $X' := X \times_Y Y'$ is naturally a G-torsor over Y', and f induces a G-equivariant morphism $X' \to X$.
For any scheme B, we define the category $[X/G](B)$ as follows.
The objects are tuples (P, π, g) where $\pi : P \to B$ is a G-torsor, and $g : P \to X$ is a G-equivariant morphism.
A morphism $(P, \pi, g) \to (P', \pi', g')$ is a G-equivariant isomorphism $\phi : P \to P'$ such that $\pi = \pi' \circ \phi$ and $g = g' \circ \phi$.

Lemma

Let X be a G-torsor over Y. Show that $Mor(B, Y) \to [X/G](B)$ defined by

$$f : B \to Y \mapsto (P := X \times_Y, \pi, g)$$

where (π, g) are the induced morphisms is an equivalence of groupoids.

Plan for final lecture

Given S an algebraic stack, G a smooth group scheme over S, and X a scheme over S with a G action, we will show that the quotient stack $[X/G]$ is algebraic and that the diagram

$$\begin{array}{ccc} G \times_S X & \xrightarrow{a} & X \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\pi]{} & [X/G] \end{array}$$

is 2-cartesian.

Given a morphism of DM type of algebraic stacks $f : X$øY with a factorisation $X \xrightarrow{i} M \xrightarrow{p} Y$ with i a closed embedding and p smooth, we define the *normal cone to f* to be $C_f := [C_{X/M}//i^* T_p]$ and show that it doesn't depend on the chosen factorisation. In fact, it can be defined even if no such factorisation exists.

We define a vector bundle stack E of rank r over X to be an algebraic stack which is locally isomorphic to a quotient $[E_1/E_0]$ where E_0 and E_1 are vector bundles on X of ranks r_0, r_1, and the action is induced by a linear map $E_0 \to E_1$.