Defining Gromov Witten invariants

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Outline of this lecture

Review of last lecture

Cotangent complex

GW obstruction space

Smoothness of $\overline{M}_{0,n}(\mathbb{P}^N, d)$

Expected dimension

GW expected dimension and virtual pullback

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- If $F : h_X \to h_Y$ is induced by a morphism of schemes $\phi : X \to Y$ then $T_f(x) = Hom(x^* \tilde{L}_\phi, \mathbb{C})$ and and $Ext^1(x^* \tilde{L}_\phi, \mathbb{C})$ is a minimal obstruction space. We will recall what $\tilde{L}_\phi$ is in the next slide.

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- If $f : X \to Y$ is smooth at $x \in X$ then it is *unobstructed*, i.e., zero is an obstruction space. The converse is also true (formal smoothness).
- If $f : X \to Y$ is étale at x then $h_{X,x}(A) \to h_{Y,y}(A)$ is a bijection for every A in (Art). The converse is also true.

Cotangent complex -1

Theorem

For a morphism of schemes $\phi : X \to Y$ which factorizes as $X \xrightarrow{i} M \xrightarrow{p} Y$, with i a closed embedding and p smooth we can set

$$\tilde{L}_\phi := i^* J \to i^* \Omega_p$$

and use it to compute T^1, T^2.

Proof.

- Statement is Zariski (and indeed étale) local in both X and Y. We may assume ϕ is affine; then it is true for any factorisation with $M = \mathbb{A}^N \times Y$.
- Given two factorizations $\phi = p_1 \circ i_1 = p_2 \circ i_2$, we can compare both to $\phi = p \circ i$ where $M := M_1 \times_Y M_2$. We get a commutative diagram:

□

Cotangent complex -2

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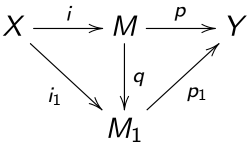

with q a smooth morphism. Let $I := \mathcal{I}_{X/M}$ and $I_1 := \mathcal{I}_{X/M_1}$.

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0 & \longrightarrow & {i_1}^* I_1 & \longrightarrow & i^* I & \longrightarrow & \Omega_q & \longrightarrow & 0 \\
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commutes and has exact, hence (since Ω_q is locally free) locally split rows. Hence it induces isomorphisms $\ker \psi_1 \to \ker \psi$ and *coker* $\psi_1 \to$ *coker* ψ, and the same is true for its arbitrary pullbacks and their duals.

Cotangent complex -3

If you know the definition of derived category, we just proved that to every morphism of schemes $\phi : X \to Y$ admitting a factorisation as a closed embeding followed by a smooth map (e.g., any morphism of quasiprojective schemes) we can associate a well defined object $\tilde{L}_\phi \in D^{-1,0}_{coh}(X)$.

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In fact, one can more generally defined the cotangent complex (Illusive,1968) $L_\phi \in D^{\leq 0}_{oh}(X)$ for an arbitrary morphism of DM type of algebraic stacks, in such a way that $\tilde{L}_\phi = \tau_{\geq -1} L_\phi$.

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The cotangent complex has a technically demanding definition and good functorial properties; it should be seen as the answer to the question "Given morphisms of schemes $h = g \circ f$, how do we extend the short exact sequence

$$f^* \Omega_g \to \Omega_h \to \Omega_f \to 0$$

to a long exact sequence?"

T^1 and T^2 for $Mor_X(C, V)$ -1

Let $p : C \to X$ and $q : V \to X$ be quasiprojective morphisms, with p flat and q smooth. Let $x_0 \in X$, and C_0, V_0 the fibres of p and q at x_0.

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$$g(A) := \{a : Spec\, A \to X \mid Im(a)_{\text{red}} = x_0\}.$$

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Lemma

$H^0(C_0, u_0^* T_{V_0})$ and $H^1(C_0, u_0^* T_{V_0})$ are tangent and obstruction space for F at u_0.

T^1 and T^2 for $Mor_X(C, V)$ -2

Proof.

Given a semismall extension $A \to B$ with kernel I, an element $a \in g(A)$ and a morphism $u_B : C_B \to V_B$, we need to prove that there is a functorial obstruction in $T^2 \otimes I$ to the existence of $u_A : C_A \to V_A$ such that $u_A|_{C_B} = u_B$. Moreover the set of such u_A (if non-empty) is a principal homogeneous space under T^1.
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Cover C_A and V_A by open affines $C_{A,i}$ and $V_{A,i}$ such that $u_0(C_{0,i}) \subset V_{0,i}$. Use previous case and Cech cohomology. □

GW obstruction space

Theorem

Let $F : \mathfrak{M}_{g,n}(V, d) \to \mathfrak{M}_{g,n}$ *be the forgetful functor. Let* $p_0 \in \mathfrak{M}_{g,n}(V, d)$ *be given by a prestable map* (C_0, x_i^0, f_0). *Then* $H^0(C_0, f_0^* T_V)$ *and* $H^1(C_0, f_0^* T_V)$ *are tangent and obstruction spaces for* F *at* p_0.

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The same holds for $\overline{M}_{g,n}(V,d) \to \mathfrak{M}_{g,n}$.

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The same holds for $\overline{M}_{g,n}(V,d) \to \mathfrak{M}_{g,n}$.

We call $H^1(C_0, f_0^* T_V)$ the *Gromov-Witten (GW) obstruction space* for F at p_0.

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Let C be a prestable curve of genus zero, L a line bundle on C such that $\deg_X L \geq 0$ for every component X of C. Then $H^1(C, L) = 0$.

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Tensor by L the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_X \oplus \mathcal{O}_Y \to \mathcal{O}_p \to 0$$

and take cohomology. □

Smoothness of $\overline{M}_{0,n}(\mathbb{P}^N, d)$

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For any n, N and d the stack $\overline{M}_{0,n}(\mathbb{P}^N, d)$ is smooth of dimension $(N+1)d + N + n - 3$.

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Smoothness Since $\mathfrak{M}_{0,n}$ is smooth, it is enough to show that the map $F : \overline{M}_{0,n}(\mathbb{P}^N, d) \to \mathfrak{M}_{0,n}$ is smooth. The obstruction space at a point $p \in \overline{M}_{0,n}(\mathbb{P}^N, d)$ corresponding to a stable map (C, x_i, f) is $H^1(C, f^* T_{\mathbb{P}^N})$. Pullback the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \bigoplus_{j=0}^{N} \mathcal{O}_{\mathbb{P}^N}(1) \to T_{\mathbb{P}^N} \to 0$$

via f, take cohomology, and note that $f^*(\mathcal{O}_{\mathbb{P}^N}(1))$ has non-negative degree on every component of X.

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Dimension The relative tangent space $T_p F$ has dimension

$$h^0(C, f^* T_{\mathbb{P}^N}) = \chi(C, f^* T_{\mathbb{P}^N}) = (N+1)d + N.$$

The fact that $\mathfrak{M}_{0,n}$ has dimension $n-3$ completes the proof.

Expected dimension -1

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The statement is unchanged if we replace X by any affine open which contains x. Hence we can assume that $X = Spec\, \mathbb{C}[x_1, \dots, x_n]/J$ and $J = (f_1, \dots, f_r)$ where $r = \dim J/\mathfrak{m}_p J$.

Expected dimension -1

Let $F : X \to Y$ be a morphism of schemes, $x \in X$, $y := f(x)$. Let $n := \dim T_x F$, and assume we are given an obstruction space $T^2_x F$ of dimension r.

Lemma Let Z be an irreducible component of X_y containing x. Then

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The expected dimension is $n - r$ and the result follows by induction on r and Lüroth's theorem.

Expected dimension -2

Definition

Let $F : X \to Y$ be a morphism of schemes. For $x \in X$ let $T^1_x F := T_x F$ the relative tangent space, and assume that at every point of $x \in X$ we are given a finite dimensional obstruction space $T^2_x F$, such that

$$d(x) := \dim T^1_x X - \dim T^2_x X \text{ is a constant } d.$$

We then say that F has *relative expected dimension* d. Note that $d \in \mathbb{Z}$ may be negative.

Expected dimension -3

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- In particular this applies if f is lci of relative dimension d, i.e., has a factorization as above such that i is a regular embedding of codimension r; then $i^*\mathcal{I}_{X/M}$ is locally free of rank r and we can choose α to be the identity.

Expected dimension -4

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It also extends to morphisms of DM type, i.e. those for which S scheme implies $X \times_S Y$ is a DM algebraic stack.

GW expected dimension and virtual pullback

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agreeing with lci pullback when F is lci of relative dimension $\bar{d}$, i.e., when the GW obstruction space is minimal at every point.
This implies we can define the *virtual fundamental class*

$$[\overline{M}_{g,n}(V,d)]^{\mathrm{vir}} := F^*_{GW}[\mathfrak{M}_{g,n}] \in A_{\bar{d}+3g-3+n}(\overline{M}_{g,n}(V,d)),$$

use it to define GW invariants and prove their properties.

The genus zero case -1

Fix $V \subset \mathbb{P}^N$, $n, d \geq 0$. as usual and let $g = 0$. Let $X = \overline{M}_{0,n}(V, d)$ and $Y = \mathfrak{M}_{0,n}$. Let $M := \overline{M}_{0,n}(\mathbb{P}^N, d)$; denote by $i : X \to M$ the natural inclusion and by $f : X \to Y$ and $p : M \to Y$ the forgetful morphisms. Recall that f and p are quasi projective and p is smooth.

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Exercise The morphism i is a closed embedding. Hint: the same is true for $\overline{M}_{g,n}(V, d) \to \overline{M}_{g,n}(W, d)$ for every closed embedding $V \to W$ of projective schemes. The analogous statement for stack of prestable maps is also true.

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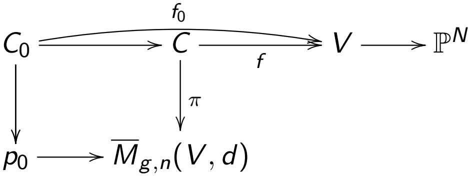
Corollary

We can define genus zero Gromov-Witten invariants.

The genus zero case -2

Sketch of the proof. Let $p_0 \in \overline{M}_{g,n}(V,d)$. Consider the diagram with cartesian square



and the exact sequence

$$0 \to T_V \to T_{\mathbb{P}^N}|_V \to N \to 0.$$

The sheaf $\mathcal{E}$ is defined to be the dual of $\pi_* f^* N$.