Defining Gromov Witten invariants

Barbara Fantechi

Chennai Mathematical Institute
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Outline of this lecture

Moduli stacks of pointed maps

$\overline{M}_{g,n}(V,d)$ is a DM algebraic stack

Properness of $\overline{M}_{g,n}(V,d)$

Summary of lecture 1

The category (Art)

Semismall extensions

Infinitesimal study of morphisms: set-up

Infinitesimal study of morphisms: lifting problem

Tangent and obstruction spaces for functors on (Art)

Moduli stacks of pointed maps -1

Definition

The *stack* $\mathfrak{M}_{g,n}(V,d)$ *of prestable, genus* g*,* n *pointed maps of degree* d is the pseudofunctor $(sch) \to (grpd)$ associating to each scheme S the groupoid of families of prestable genus g, n-pointed maps over S with their isomorphisms.

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We denote by $F : \mathfrak{M}_{g,n}(V,d) \to \mathfrak{M}_{g,n}$ the *forgetful morphism*, mapping a family of stable maps to its family of prestable curves and forgetting the map.

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We also denote by F the restriction of the forgetful morphism to $\overline{M}_{g,n}(V,d)$.

Moduli stacks of pointed maps-2

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The stack $\mathfrak{M}_{g,n}(V, d)$ *is an (Artin) algebraic stack, locally of finite type over* $\mathbb{C}$*.*

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Let S be a scheme and $S \to \mathfrak{M}_{g,n}$ a morphism, i.e., a family of prestable genus g, n-pointed curves. We need to prove that the fiber product $S \times_{\mathfrak{M}_{g,n}} \mathfrak{M}_{g,n}(V,d)$ is a scheme, quasiprojective over S.

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It follows easily from the definition of fiber product for stacks that it is isomorphic to $Mor_S(C, V \times B)^d$. □

Algebraicity of $\overline{M}_{g,n}(V,d)$ -1

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1. *The stack $\overline{M}_{g,n}(V,d)$ is a Deligne-Mumford (DM) algebraic stack.*
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$\overline{M}_{g,n}(V,d)$ is an algebraic stack because it is open in $\mathfrak{M}_{g,n}(V,d)$ which is algebraic; in particular, the forgetful morphism is quasi projective. □

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By assumption the automorphism group is finite; since we are in characteristic zero, it is smooth, hence its tangent space is zero-dimensional.

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This argument fails in positive characteristic, and indeed in that case the stack is not DM in general. □

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Let $\overline{B}$ be any smooth affine curve, $b_0 \in \overline{B}$ any point, and $B = \overline{B} \setminus b_0$.
We need to show that any family of stable maps (C, π, x_i, f) over B can be uniquely extended to $\overline{B}$, after possibly a finite base change.

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We first use a base change to extend (C, π, x_i) to $(\overline{C}, \bar{\pi}, \bar{x}_i)$ over $\overline{B}$.
This can be done in analogy with the proof of properness for $\overline{M}_{g,n}$.

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Assume for simplicity that $\overline{C}$ is a smooth surface. Then $f : C \to V$ induces a rational map $\overline{C} \to V$; after a finite number of blow-ups $\varepsilon : \overline{C}' \to \overline{C}$, we can assume that the map $f' := f \circ \varepsilon$is regular.

Properness of $\overline{M}_{g,n}(V,d)$ -2

The fibres of $\bar{\pi} \circ \varepsilon : \overline{C}' \to \overline{B}$ are nodal curves but may be non reduced. This can be fixed by a finite base change and normalisation.

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We have thus proven that, up to base change, we can extend (C, π, x_i, f) to a family of prestable maps over $\overline{B}$. We now need to show that among such extensions there is a unique stable one.

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First we prove existence. Let $(\overline{C}, \bar{\pi}, \bar{x}_i, \bar{f})$ be a prestable extension to B. If it isn't stable, there is a rational curve Z in C_{b_0}, contracted by $\bar{f}$, whose normalisation $\widetilde{Z}$ contains at most two special points.

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The same argument applies when there are two special points, one a node and one marked.

Properness of $\overline{M}_{g,n}(V, d)$ -3

Proof.

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Again by connectedness, Z must also be smooth, and one can prove that $N_{Z/C} = \mathcal{O}_Z(-2)$. Hence Z can be contracted as above, but this time to an A_1 singularity.

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Repeating the process, we get in the situation we had before, except now $\overline{C}'$ has rational double points. This proves existence.

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Proof.

To prove uniqueness, any other extension must be birational to the one we started with; if they are both smooth, a birational map factors uniquely as a sequence of blow-ups and blow-downs.
One can prove by induction on the total number of bloe-ups and blow-downs that the birational map must be an isomorphism.

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For the general case, we cannot assume that $\overline{C}$ is smooth but its singularities are very limited, either nodes

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or A_n

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One can extend the previous argument to this case, working with the minimal resolution of singularities of $\overline{C}$, which is easy to construct explicitly. □

Summary of the previous lecture

For any projective smooth variety (indeed, any projective scheme) V we have defined a proper DM algebraic stack of (families of) stable maps $\overline{M}_{g,n}(V,d)$; by definition it carries a universal genus g, n-pointed stable map (C,π,x_i,f) of degree d.

The forgetful morphism $F : \overline{M}_{g,n}(V,d) \to \mathfrak{M}_{g,n}$ is quasiprojective.

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1. A similar argument with minimal models of surfaces proves the properness of $\overline{M}_{g,n}$.
2. Replacing minimal models with semistable reduction gives properness of $\overline{M}_{g,n}$ and $\overline{M}_{g,n}(V,d)$ in any characteristic.
3. However while $\overline{M}_{g,n}$ is DM in any characteristic (and indeed over $\mathbb{Z}$), $\overline{M}_{g,n}(V,d)$ is not.

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We denote by (Art) the category of local $\mathbb{C}$ algebras satisfying the equivalent conditions above.

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Note that $\mathbb{C}$ is both an initial and a final object in (Art).

Semismall extensions -1

Definition

Let $\phi : A \to B$ be a surjective morphism in (Art), $I = \ker \phi$. The exact sequence

$$0 \to I \to A \to B \to 0 \qquad (1)$$

is called a *semi-small extension* if $I \cdot \mathfrak{m}_A = 0$.
We also say that ϕ or *Spec* $B \to$ *Spec* A is a semismall extension.

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Every surjective morphism in (Art) (or equivalently, every closed embedding of fat points) factors as a finite sequence of semismall extensions.

Semismall extensions -2

Definition

A *morphism* of semismall extensions is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & I & \to & A & \to & B & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I' & \to & A' & \to & B' & \to & 0 \end{array}$$

where rows are semismall extensions and $A \to A'$ is a morphism in (Art).

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Exercise Show that this implies $B \to B'$ is a morphism in (Art).

Set-up

Let X be a scheme, and h_X : (Art) $\to$ (sets) the (covariant) functor

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Let $\widehat{\mathcal{O}}_{X,x} := \lim \mathcal{O}_{X,x}/\mathfrak{m}_x^N$. There is a natural, functorial bijection

$$h_{X,x} \to Hom_{alg}(\widehat{\mathcal{O}}_{X,x}, A)$$

Set-up - 2

Let $f : X \to Y$ be a morphism of schemes, $x \in X$, $y = f(x)$.
Let $R_y := \widehat{\mathcal{O}}_{Y,y}$ and $R_x := \widehat{\mathcal{O}}_{X,x}$;
let $f^* : R_y \to R_x$ be the homomorphism induced by $(f, f^\sharp)$.
Let $A \to B$ be a semismall extension with kernel I.

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Assume we are given a commutative diagram of schemes

$$\begin{array}{ccc} Spec\, B & \longrightarrow & X \\ \downarrow & & \downarrow \\ Spec\, A & \longrightarrow & Y \end{array}$$

with $Im(Spec\, B)_{\text{red}} = x$.
We want to study the set of morphisms $Spec\, A \to X$ making the diagram commute (infinitesimal lifting problem).

Set-up -3

Equivalently, we have a commutative diagram of local algebras

$$\begin{array}{ccc} R_y & \xrightarrow{\alpha_Y} & A \\ \downarrow & & \downarrow \\ R_x & \xrightarrow{\beta} & B \end{array}$$

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WLOG assume that X and Y are affine of f.t. over $\mathbb{C}$.
Hence $X \to Y$ factors (non-canonically) as

$$X \xrightarrow{i} \mathbb{A}^n \times Y \xrightarrow{p} Y.$$

with i a closed embedding.

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with i a closed embedding. Therefore $R_y \to R_x$ factors as

$$R_y \to R_y[[t_1, \ldots, t_n]] \xrightarrow{i^*} R_x$$

with i^* surjective; let $J = \ker i^*$.

Lifting problem -1

Lemma

Let $S := R_y[[t_1, \ldots, t_n]]$*. The set of liftings* $\alpha_S : S \to A$ *such that the diagram*

$$\begin{array}{ccc} R_y & \xrightarrow{\alpha_Y} & A \\ \downarrow & & \downarrow \\ S & \xrightarrow{\beta \circ i^*} & B \end{array}$$

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Proof.

(Sketch) It is easy to show that a lifting exists (take the image of each t_i in B and lift it to A). Show that the difference of two liftings is a map $\lambda : S \to I$ which is an R_y-derivation, and conversely.

Hence the set of liftings is a p.h.s. for $Der_{R_y}(S, I)$. □

Lifting problem -2

Lemma

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Proof.

$\square$

Lifting problem -2

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Proof.

$$Der_{R_y}(S, I) = Hom_S(\Omega_{S/R_y}, I) = Hom_{S/\mathfrak{m}_s}(\Omega_p(x), I)$$

since $\mathfrak{m}_S \cdot I = 0$. Here $\Omega_p(x) := x^*\Omega_p$.

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By construction the natural map $x^*\Omega_p \to x^*\Omega_f$ is an isomorphism. □

Lifting problem -3

Let us denote by L the set of liftings $S \to A$ as before. Fix a lifting $\phi_0 \in L$ as before. We get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & J & \to & S & \to & R_x & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I & \to & A & \to & B & \to & 0 \end{array}$$

There is a morphism $\alpha : R_x \to A$ making the diagram commute if and only if $\lambda_0 := \phi_0|_J : J \to I$ is zero; in this case, α is unique. I.e., we have a natural map $\rho : L \to Hom_{\mathbb{C}}(J/\mathfrak{m}_s J, I)$ such that the set of liftings is the inverse image of zero.

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Claim The map ρ is equivariant for the action of $Hom_{\mathbb{C}}(\Omega_p(x), I)$, induced by the exact sequence on X

$$i^* I_{X/\mathbb{A}^n \times Y} \to i^* \Omega_p \to \Omega_f \to 0$$

via x^* together with the fact the fact that $x^* I_X = x^* J$.

Lifting problem -4

Definition

We define $Hom(x^*\tilde{L}_f, I)$ and $Ext^1(x^*\tilde{L}_f, I)$ as kernel and cockernel of the map

$$Hom(x^*\Omega_p, I) \to Hom(x^*i^*I_X, I).$$

Theorem

Let $h = h_{X,x}$ and $g := h_{Y,,y}$. We have shown that there is an exact sequence of groups and sets

$$0 \to Hom(x^*\tilde{L}_f, I) \to h(A) \to h(B) \times_{g(B)} g(A) \stackrel{ob}{\to} Ext^1(x^*\tilde{L}_f, I).$$

Here a sequence of groups and sets

$$0 \to A_0 \xrightarrow{a} Z_0 \xrightarrow{\psi} Z_1 \xrightarrow{ob} A_1$$

with A_i abelian groups and Z_i sets, is called *exact* if a is an action of A_0 on X_0 which acts simply transitively on the fibres of ψ, and ob is a set map such that $ob(z_1) = 0$ if and only if $\psi^{-1}(z_1) \neq \emptyset$.

Lifting problem -5

Here the map *ob* is defined as follows: given an element z_1 in $h(B) \times_{g(B)} g(A)$, i.e., a commutative diagram, choose a lifting $\phi_0 : S \to A$ and let $ob(z_1)$ be its image in $Ext^1(x^*\tilde{L}_f, I)$.

Theorem

The exact sequence of groups and sets is functorial in semismall extensions, i.e. a morphism of $A \to B$ to $A' \to B'$ induces a commutative diagram

$$\begin{array}{ccccccc} Hom(x^*\tilde{L}_f, I) & \to & h(A) & \to & h(B) \times_{g(B)} g(A) & \stackrel{ob}{\to} & Ext^1(x^*\tilde{L}_f, I) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Hom(x^*\tilde{L}_f, I') & \to & h(A') & \to & h(B') \times_{g(B')} g(A') & \stackrel{ob}{\to} & Ext^1(x^*\tilde{L}_f, I'). \end{array}$$

Tangent and obstruction spaces -1

Let $\bar{f}, \bar{g} : (\mathrm{Art}) \to (\mathrm{sets})$ be functors, and $\bar{F} : \bar{f} \to \bar{g}$ a natural transformation. Choose $x \in \bar{f}(\mathbb{C})$ and let $y = \bar{F}(x) \in g(\mathbb{C})$. Let $f := \bar{f}_x : (\mathrm{Art}) \to (\mathrm{sets})$ be the fiber functor

$$f(A) := \{a \in \bar{f}(A) \,|\, \pi_{A*}(a) = x\} \quad \text{where } \pi_A : A \to \mathbb{C},$$

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Definition

Let $T^1_x\bar{F}$, $T^2_x\bar{F}$ be $\mathbb{C}$ vector spaces. We say that they are the tangent space and an obstruction space for $\bar{F}$ at x if for any semismall extension $A \to B$ there is a functorial exact sequence

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Note the difference between **the** tangent space and **an** obstruction space.

Tangent and obstruction spaces -2

Lemma

A tangent space, if it exists, is unique up to canonical isomorphism and can be identified with the fiber over the image of $y \in g(\mathbb{C})$ *of the map* $f(\mathbb{C}[\varepsilon]/\varepsilon^2) \to g(\mathbb{C}[\varepsilon]/\varepsilon^2)$.

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The obstruction space cannot be unique, since any vector space containing it is also an obstruction space. An obstruction space is *minimal* if all others can be obtained in this way; a minimal obstruction space, if it exists, is unique up to canonical isomorphism.

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Lemma

Let $\phi : X \to Y$ be a morphism of schemes, $\bar{F} : h_X \to h_Y$ the induced natural transformation. Then $Hom(x^*\tilde{L}_f, \mathbb{C})$ is the tangent space and $Ext^1(x^*\tilde{L}_f, \mathbb{C})$ is the minimal obstruction space for $\bar{F}$ at x.

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We also speak of tangent and obstruction spaces to f at x.

Tangent and obstruction spaces: exercises

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3. Let

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

   be a cartesian diagram of schemes, $x' \in X'$ and $x \in X$ its image. Show that if T^1 and T^2 are tangent and obstruction space for f at x, then they are also tangent and obstruction space for f' at x'.

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4. If T^2 is minimal for f at x it may not be minimal for f' at x'; it is minimal if $Y' \to Y$ is smooth.