The decidability frontier for Petri nets

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Petri nets

- A set $P$ of places
- A set $T$ of transitions
- Flow relation $F : (P \times T) \cup (T \times P) \rightarrow \mathbb{N}_0$
- Initial marking $M_0 : P \rightarrow \mathbb{N}_0$
- Dynamics: "Token game"
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![Petri net diagram]
Petri nets

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Decision questions

- **Reachability**
  Is a marking $M$ (exactly) reachable from $M_0$?

- **Coverability**
  Is a marking $M$ coverable from $M_0$?
  - Can we reach $M'$ such that for each $p$, $M'(p) \geq M(p)$

- **Termination**
  Is there an infinite execution?

- **Boundedness**
  Is the set of reachable markings finite
  - Is there a bound $B$ such that no place has more than $B$ tokens in any reachable marking?

- **Place-boundedness**
  For a given place $p$, is the number of tokens on $p$ bounded in all reachable markings?
Decision questions . . .

- All these questions are decidable for “normal” Petri nets
  - Some proofs are easy (boundedness), others less so (reachability)
  - Classifying the computational complexity is a separate issue that we will not discuss
Decision questions . . .

Inhibitor arcs

- Petri net places are like counters without test-for-zero
Decision questions . . .

Inhibitor arcs

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- If we can check for the absence of tokens, everything becomes undecidable

\[ t \] is enabled at \( M \) only if \( M(p_1) > 0 \) and \( M(p_2) = 0 \)

- Two inhibitor arcs can simulate 2 counter machine
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\[
\begin{align*}
  p_1 & \quad t & \quad p_3 \\
  p_2 & \quad t & \quad p_3
\end{align*}
\]

- \( t \) is enabled at \( M \) only if \( M(p_1) > 0 \) and \( M(p_2) = 0 \)
- Two inhibitor arcs can simulate 2 counter machine
Boundedness

Karp-Miller reachability tree

- Start with the initial marking $M_0$
- Use BFS through space of reachable markings
  - Let $M$ be a leaf node with $t$ enabled at $M$ such that $M \xrightarrow{t} M'$
  - Add $M'$ as a new leaf if it does not already appear on the path from $M_0$ to $M$
- Acceleration
  - If $M' > M''$ for some marking on the path from $M_0$ to $M$, set $M'(p) = \omega$ wherever $M'(p) > M''(p)$
A marking \( M \) over \( k \) places is a vector over \( \mathbb{N}^k \). Given any infinite sequence of markings \( M_1, M_2, \ldots \), there must exist positions \( i \) and \( j \) such that \( i < j \) and \( M_i \leq M_j \).
Dickson’s lemma

A marking $M$ over $k$ places is a vector over $\mathbb{N}^k$

Given any infinite sequence of markings $M_1, M_2, \ldots$, there must exist positions $i$ and $j$ such that $i < j$ and $M_i \leq M_j$

- Cannot have an infinite set of incomparable markings
The Karp-Miller tree

Boundedness and termination are decidable

The Karp-Miller tree is always finite, by Dickson’s Lemma.

The given net is bounded iff $\omega$ does not appear in the tree.

The given net terminates if we can always expand all transitions fully in the tree.

- Never repeat a marking on any path
- Never apply acceleration
The Karp-Miller tree

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The Karp-Miller tree, in fact, decides place-boundedness
Coverability

- For a set of markings $S$, $\text{Pred}(S)$ is the set of markings from where we can reach $S$.

- If $S$ is upward-closed, so is $\text{Pred}(S)$.

- Any upward closed set $S$ has a finite set of minimal elements $\{s_1, s_2, \ldots, s_k\}$ such that $S = \uparrow\{s_1, s_2, \ldots, s_k\}$—finite basis for $S$.

- The set of markings that cover $M$ is upward closed.

- Iteratively compute a finite basis for $\text{Pred}(\uparrow M)$. 
What makes Petri net properties decidable?

- A set of incomparable markings must be finite
- Firing rule is compatible with marking order:

\[
\begin{align*}
M & \xrightarrow{t} M' \\
\land \\
M_1
\end{align*}
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- A set of incomparable markings must be finite
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  \[
  M \xrightarrow{t} M'
  \]
  \[
  \land M_1 \xrightarrow{t} M_1'
  \]

- In fact \((M_1 - M) = (M_1' - M')\)
- Thus, \(M < M_1\) implies \(M' < M_1'\) — strict monotonicity
Well structured transition systems

Well quasi-order (wqo)

- $(X, \preceq)$, $\preceq$ is reflexive and transitive
- Given any infinite sequence $x_1, x_2, \ldots$ over $X$, there must exist positions $i$ and $j$ such that $i < j$ and $x_i \preceq x_j$
Well structured transition systems

Well quasi-order (wqo)

1. $(X, \preceq)$, $\preceq$ is reflexive and transitive
2. Given any infinite sequence $x_1, x_2, \ldots$ over $X$, there must exist positions $i$ and $j$ such that $i < j$ and $x_i \preceq x_j$

Note that this also rules out infinite descending chains.
Well structured transition systems

Well quasi-order (wqo)

\((X, \preceq), \preceq\) is reflexive and transitive

Given any infinite sequence \(x_1, x_2, \ldots\) over \(X\), there must exist positions \(i\) and \(j\) such that \(i < j\) and \(x_i \preceq x_j\)

Note that this also rules out infinite descending chains.

\((X, \rightarrow)\) is a well structured transition system if there is exists a wqo \((X, \preceq)\) such that \(\rightarrow\) is compatible with \(\preceq\)

\[
\begin{align*}
\text{x} & \rightarrow \text{x}' \\
\gamma! & \rightarrow \gamma! \\
x_1 & \rightarrow x_1'
\end{align*}
\]
Well structured transition systems . . .

Concrete decision procedures for Petri nets can be lifted to WSTSSs

- Karp-Miller tree generalize to finite reachability tree

- Can use this to decide termination
Well structured transition systems . . .

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  - For boundedness, we need strict monotonicity

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\begin{align*}
  x & \rightarrow x' \\
  x_1 & \rightarrow x'_1
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- Backward saturation to compute coverability if the WSTS has an effective pred-basis

Given a state \( x \in X \), compute a finite basis for \( \text{Pred}(\uparrow x) \)
Generalized Petri nets

- Petri net with arc weights labelled by polynomials over places
  - Evaluate polynomial with respect to current marking
  - Resulting value determine whether a transition is enabled . . .
  - . . . and computes the effect of firing it.
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```
p_1  t_1  p_2  t_2  p_3
  p_3   2*p_2  p_1
```
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```
p_1
  \rightarrow t_1
  \rightarrow p_2
  ^\  2 \cdot p_2
  \rightarrow t_2
  \rightarrow p_3
  \rightarrow p_1
```
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  - ... and computes the effect of firing it.

- Fibonacci net
  For odd $k$, marking $m_k = (\text{fib}(k+1), 0, \text{fib}(k))$
Generalized nets ... 

- All problems are undecidable in general
  - Subsume inhibitor arcs
  - To fire $t_1$, we need $2 \cdot M(p2)$ tokens at $p_2$
  - $M(p_2)$ must be 0!

- Subclasses clearly separate decision boundaries for reachability, coverability, termination, boundedness, place boundedness,
Decision problems for reset post-G nets

- **Reset arc**: \( W(p, t) = p \)
  - Resets (i.e., empties) input place \( p \) when \( t \) fires

- **Transfer arc**: \( W(p, t) = p = W(t, p') \)
  - Transfers contents of \( p \) to \( p' \)

- **Post-G net**: only output arcs are non-classical

- **Double Petri net**: Post G-net where \( F(t, p) = p \) or \( F(t, p) \in \mathbb{N} \).
  - \( F(t, p) = p \): doubling arc: doubles the marking of \( p \)
Reset Post-G nets

- Input arcs are either reset or classical
- Output arcs can have arbitrary polynomials
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What’s decidable
Coverability, termination

Reset post-G nets define WSTSs with effective pred-basis, but do not satisfy strict monotonicity.
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**What’s decidable**

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Reset post-G nets define WSTSSs with effective pred-basis, but do not satisfy strict monotonicity.

**What’s not**

Boundedness

Reset post-G nets can “compute” polynomials. Complicated reduction from Hilbert’s Tenth Problem.
Transfer nets

- All non-classical arcs are pairs that define transfers
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- Simulate a reset post-G net $N$ by a transfer net $N'$.  
- Add a dummy place to $N$ to get $N'$. Simulate resets by transferring tokens to this dummy place.  
- $N$ is unbounded iff some place other than the dummy place is unbounded in $N'$. 
Post-G nets

- Input arcs are classical, only output arcs have extended weights

What’s decidable

Place-boundedness

Post-G nets define WSTSs with strict monotonicity and an additional continuity condition required to compute place boundedness from the finite reachability tree.
Reachability

Undecidability

Reachability is undecidable for double Petri nets, reset Petri nets and transfer Petri nets with two extended arcs.

Two extended arcs can simulate nets with inhibitor arcs.
What’s decidable?
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