Model-Checking Event Structures, Part 2

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Concurrent systems

- Convenient to view each execution as a labelled partial order
Mazurkiewicz traces

- Actions are enriched with independence relation specifying which pairs are independent
  - Symmetric, irreflexive
  - Typically derived from structure of underlying system
    - Actions performed by disjoint sets of components
- In a linearization, adjacent independent actions can be swapped to yield an equivalent linearization
From traces to event structures

- Can extract an event structure from the set of traces
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  - For instance, \([e_1 e_2 e_3] \leq [e_1 e_4 e_2 e_3]\)
Can extract an event structure from the set of traces

- $t \leq t'$ if $t'$ extends $t$ with more events
  - For instance, $[e_1 e_2 e_3] \leq [e_1 e_4 e_2 e_3]$

- $t$ and $t'$ are compatible if there is $t''$ such that $t \leq t''$ and $t' \leq t''$
  - For instance, $[e_1 e_2 e_3]$ and $[e_4]$ are compatible because both are dominated by $[e_1 e_2 e_3 e_4]$
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- \( t \# t' \) if \( t \) and \( t' \) are not compatible
From traces to event structures

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  - For instance, \([e_1 e_2 e_3]\) and \([e_4]\) are compatible because both are dominated by \([e_1 e_2 e_3 e_4]\)
- \( t \neq t' \) if \( t \) and \( t' \) are not compatible
- Identify **events with prime traces**
  - **Prime trace**: Only one maximal element
  - “Earliest” occurrence of an action
Event Structures . . .
Formally, an event structure is of the form $ES = (E, \leq, \# , \lambda)$

- $E$ is the set of event occurrences
- $\leq$ is the causality relation (a partial order)
- $\#$ is a binary conflict relation
  - Irreflexive, symmetric
- Conflict is inherited via causality
  - $e \# f$ and $f \leq f'$ implies $e \# f'$
- $\lambda : E \rightarrow \Sigma$ labels each event occurrence with an action
- Two events are concurrent if they are not related by $\leq$ or $\#$
  — $e \text{ co } f$
Trace event structures

Let \((\Sigma, I)\) be a trace alphabet

\[ ES = (E, \leq, \#, \lambda) \] is a trace event structure if

- \(e \#_\mu f \Rightarrow \lambda(e) \neq \lambda(f)\)
  - Determinacy!

- If \(e < f\) or \(e \#_\mu f\), \((\lambda(e), \lambda(f)) \notin I\)

- If \((\lambda(e), \lambda(f)) \notin I\) then \(e \leq f\) or \(f \leq e\) or \(e \# f\).
Trace event structures

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$ES = (E, \leq, \#, \lambda)$ is a trace event structure if

- $e \#_\mu f \Rightarrow \lambda(e) \neq \lambda(f)$
  - Determinacy!

- If $e \lessdot f$ or $e \#_\mu f$, $(\lambda(e), \lambda(f)) \notin I$

- If $(\lambda(e), \lambda(f)) \notin I$ then $e \leq f$ or $f \leq e$ or $e \# f$.

Fact

Any event structure constructed from the traces of a deterministic concurrent system is a trace event structure
Event structures as relational structures

Instead of temporal logics, consider

- First-Order Logic (FOL)
- (Variations of) Monadic Second Order logics (MSO)

FOL and MSO are logics over relational structures — a set with a collection of relations defined over the set

Labelled event structures give rise naturally to relational structures

- \( ES = (E, \leq, \#, \lambda) \) labelled by \( \Sigma = \{a_1, a_2, \ldots, a_n\} \)

- Corresponding relational structure is \( (E, \leq, \#, \ell_{a_1}, \ell_{a_2}, \ldots, \ell_{a_n}) \)
  - Each \( \ell_{a_i} \) is a unary predicate such that \( \ell_{a_i}(e) \) is true iff \( \lambda(e) = a_i \)
FOL and MSO

Relational structure \((E, \leq, \#, \ell_a_1, \ell_a_2, \ldots, \ell_a_n)\)

- \(\{x, y, \ldots\}\) : variables representing individual events
- \(\{X, Y, \ldots\}\) : variables representing sets of events

FOL

\[ x = y | x \leq y | x \# y | \ell_a(x) | \neg \varphi | \varphi \land \varphi | \exists x. \varphi(x) \]

MSOL

\[ x = y | x \leq y | x \# y | \ell_a(x) | \neg \varphi | \varphi \land \varphi | \exists x. \varphi(x) | \exists X. \varphi(x) \]
The model-checking problem

- We are given a regular trace language $L$
  - Set of traces whose linearizations is a regular language
- From the prime traces, those with a single maximal event, we can extract an event structure $ES_L$
- Given a formula $\varphi$ in FOL/MSO, does $ES_L \models \varphi$?
Alphabet \( \{a, b, c\} \) with \( l = \{(a, b), (b, a)\} \)

Consider trace language generated by words of the form \( a^* b^* c \)

Each prime trace/event \([a^j b^k c]\) encodes a grid point \((j, k)\)

Set variables describe an assignment of colours to these events

MSO can describe that this colouring/tiling of the grid is valid

To get around this, restrict MSO to Monadic Trace Logic (MTL)

Quantify over conflict-free subsets of \( E \)
FOL over trace event structures is decidable

- Let $\varphi(x_1, x_2, \ldots, x_k)$ be an FOL formula
  - $\varphi$ defines a $k$-ary relation over events
    \[ R_\varphi = \{(e_1, e_2, \ldots, e_k) : ES \models \varphi(e_1, e_2, \ldots, e_k)\} \]
- Recall that each event is actually a prime trace, so $R_\varphi$ is a relation over traces in $L$
- Combine each tuple $(t_1, t_2, \ldots, t_k) \in R_\varphi$ into a single braided trace (over a new alphabet)
- Model-checking $R_\varphi$ is equivalent to checking that the set of braided traces corresponding to $R_\varphi$ is non-empty
- For each formula $\varphi$, the braided traces corresponding to $R_\varphi$ form a regular trace language
Overlap traces as far as possible, recording for each overlapped event, which components participate in that event.
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\[
\begin{align*}
(e_1, e'_1) & : a, \{x_1, x_2\} \\
(e_2, e'_2) & : b, \{x_1, x_2\} \\
(e_3, e'_3) & : c, \{x_1, x_2\} \\
e_4 & : a, \{x_1\} \\
e'_4 & : b, \{x_2\} \\
e'_5 & : c, \{x_2\}
\end{align*}
\]
Braiding traces...

- Braided traces over new alphabet $\Sigma_B$ with symbols $(a, Y)$ where
  - $a \in \Sigma$ is a letter from the original alphabet
  - $Y \subseteq \{x_1, x_2, \ldots, x_k\}$

- $((a, X), (b, Y)) \in I_B$ if $(a, b) \in I$ or $X \cap Y = \emptyset$
Braiding traces . . .

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Observation

- If $(a, X) \leq (b, Y)$ in a braided trace, then $Y \subseteq X$
  - The second component monotonically decreases along each chain of dependent letters
- This property can be checked by a finite-state automaton
Braiding traces . . .

Theorem

For each FOL formula $\varphi(x_1, x_2, \ldots, x_k)$, the corresponding braided trace language is regular

Proof

By induction on the structure of $\varphi$
Braiding traces . . .

\( \varphi \) is \( x = y \)
Braiding traces . . .

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- \((t_1, t_2) \in R_\varphi\) iff \( t_1 = t_2 \)
- Braided trace is isomorphic to \( t_1 \) (and \( t_2 \))
- Each action is labelled \( \{x_1, x_2\} \)
Braiding traces . . .

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- Braided trace is isomorphic to \( t_1 \) (and \( t_2 \))
- Each action is labelled \( \{x_1, x_2\} \)
- Check that projection onto \( \Sigma \) is a prime trace in \( L \)
  - Note: If \( L \) is a regular trace language, the prime traces of \( L \) also form a regular trace language
- Check that second component of each label is \( \{x_1, x_2\} \)
Braiding traces . . .

\( \varphi \) is \( x \leq y \)
Braiding traces ...

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- \((t_1, t_2) \in R_\varphi\) iff \( t_2 \) extends \( t_1 \)
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Braiding traces . . .

ϕ is \#y
Braiding traces . . .

\( \varphi \) is \( x \neq y \)

- \( (t_1, t_2) \in R_\varphi \) iff \( t_1 \) and \( t_2 \) diverge
- At least one action each labelled only \( \{x_1\} \) and \( \{x_2\} \)
- Braided trace restricted to
  - actions labelled \( \{x_1, x_2\} \) or \( \{x_1\} \) is isomorphic to \( t_1 \)
  - actions labelled \( \{x_1, x_2\} \) or \( \{x_2\} \) is isomorphic to \( t_2 \)
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  - actions labelled \( \{x_1, x_2\} \) or \( \{x_2\} \) is isomorphic to \( t_2 \)

- Check that projections \( \{x_1, \ldots\} \) and \( \{x_2, \ldots\} \) are both prime traces in \( L \)

- Check that there is at least one event each with second component of label \( \{x_1\} \) and \( \{x_2\} \)

- Check that second component decreases monotonically along each chain of dependent letters
Braiding traces . . .

\( \varphi \) is \( \exists y. \psi(y, x_1, \ldots, x_k) \)

- By induction hypothesis, braided trace language for \( R_\psi \) is regular
- Define a natural projection operator to eliminate \( y \) from a set of braided traces
  - Project onto \((x_1, \ldots, x_k) \Rightarrow \) drop \( y \) from each event’s label
  - Erase any event whose initial label was \( \{y\} \) (and hence now has an empty label)
- If \( B \) is a regular language of braided traces over variables \( \bar{x} \), its projection onto any subset of \( \bar{x} \) is also regular
- Braided trace language for \( \varphi \) is obtained by projecting the language for \( \psi \) onto \((x_1, x_2, \ldots, x_k) \)
Braiding traces . . .

ϕ is ¬ψ : Easy

ϕ is ψ₁ ∧ ψ₂

► ψ₁(x₁, . . . , xₖ) and ψ₂(y₁, . . . , yₘ) so braided traces for ϕ are over (x₁, . . . , xₖ, y₁, . . . , yₘ)

► In general, some variables overlap between ψ₁, ψ₂

ϕ(x, y, z) = ψ₁(x, z) ∧ ψ₂(y, z)

► Define an “expansion” operator:

► B, a set of braided traces over u = (u₁, u₂, . . . , uₖ)

► v = (v₁, v₂, . . . , vₘ), a new set of variables

► B ↑ v : all braided traces over (u, v) = (u₁, . . . , uₖ, v₁, . . . , vₘ) whose projection onto u lies in B.

► Then, the language for ϕ is (Bψ₁ ↑ y) ∩ (Bψ₂ ↑ x)
MTL

- MTL is MSO with set quantifiers restricted to conflict-free subsets of $E$
- In FOL proof, each individual variable $x$ is assigned an event $e$, which can be regarded as a prime trace
- Can we represent conflict-free subsets of $E$ as traces?
MTL

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- In FOL proof, each individual variable \( x \) is assigned an event \( e \), which can be regarded as a prime trace.
- Can we represent conflict-free subsets of \( E \) as traces?
- If \( X \subset E \) is conflict-free, so is \( \downarrow X \).
- Thus, \( \downarrow X \) is a trace (not necessarily prime).
- Not all events in \( \downarrow X \) are part of the subset.
  - Add a tag from \( \{\perp, \top\} \) to indicate which events in \( \downarrow X \) belong to \( X \) and which do not.
MTL

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- Can we represent conflict-free subsets of $E$ as traces?
- If $X \subset E$ is conflict-free, so is $\downarrow X$
- Thus, $\downarrow X$ is a trace (not necessarily prime)
- Not all events in $\downarrow X$ are part of the subset
  - Add a tag from $\{\bot, \top\}$ to indicate which events in $\downarrow X$ belong to $X$ and which do not
- Can again assign a set of braided traces with each formula $\varphi$
- Show by induction on $\varphi$ that this set is regular
In perspective

FOL over traces can express all natural temporal modalities

- $ES, e \models A_{\leq} \varphi$ if at every $f$ such that $e \leq f$, $ES, f \models \varphi$
- $ES, e \models E_{\#} \varphi$ if there exists $f$ such that $e \not\# f$ and $ES, f \models \varphi$
- $ES, e \models A_{co} \varphi$ if at every $f$ such that $e \ co f$, $ES, f \models \varphi$
- ...
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- \( ES, e \models A_{co} \varphi \) if at every \( f \) such that \( e \ co f \), \( ES, f \models \varphi \)
- \( \ldots \)

In one shot, decidability of FOL over trace event structures shows that all (reasonable) temporal logics are decidable!
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- ... 

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What more remains to be done?
In perspective . . .

- System is presented as a regular trace language

- Implicitly, we assume a deterministic machine recognizing the language

- Model-checking is typically applied to a given system model
  - May be nondeterministic
  - Distinction between labelled and unlabelled systems in models like Petri nets

- What is the status of branching-time model-checking for labelled concurrent systems?
In sequential systems, model-checking is intimately connected to automata theory

- Tree automata
- Alternating automata (on strings and trees)

In concurrent systems, the theory of “string” automata is reasonably well-understood

- Asynchronous automata, Zielonka’s theorem

How do we define alternating automata on traces?