Infinite graphs with decidable MSO theories

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  - Example: $\forall x \, \exists y \; x < y$, $\forall x \, \exists y \; y < x$

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    Principle of mathematical induction

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  - φ is satisfiable iff $L(M_\varphi)$ is nonempty

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  - All follow by MSO interpretations in S2S

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$\exists X \varphi(X)$ in S3S $\mapsto$ $\exists X(X \subseteq T \wedge \tilde{\varphi}(X))$ in S2S

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Proposition If $\mathcal{A}$ is MSO-interpretable in $\mathcal{B}$ and MSO is decidable over $\mathcal{B}$ then MSO is decidable over $\mathcal{A}$

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  Unfolding of G_0 is the binary tree T_2

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- Theorem also holds for a different type of unfolding called tree iteration

  Due to [Muchnik (reported by Semenov 1985)] and [Walukiewicz 2002]

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The Caucal hierarchy, by example

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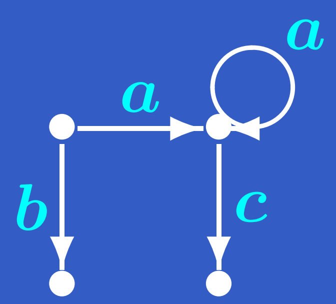

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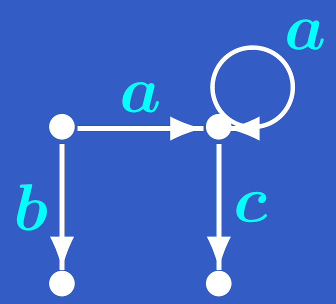

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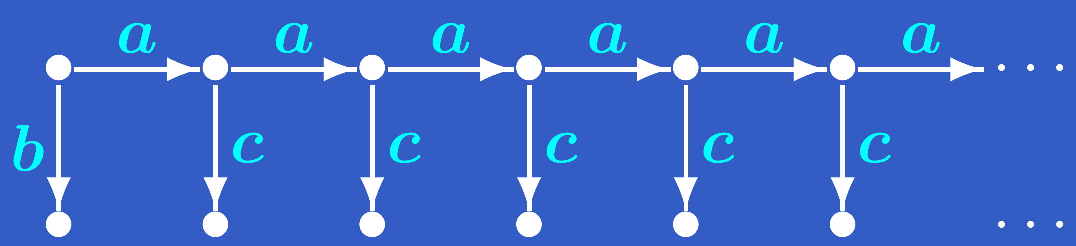

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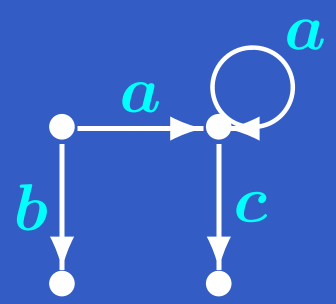

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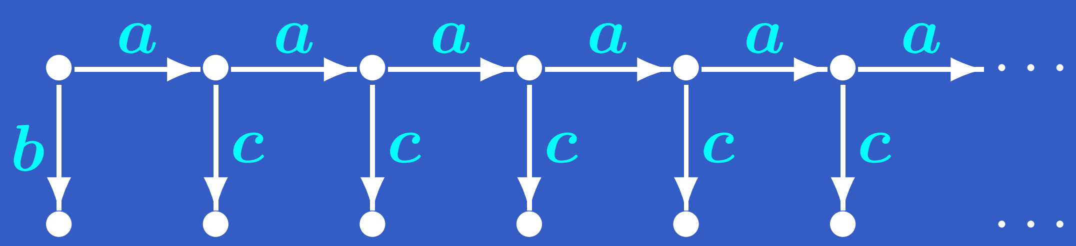

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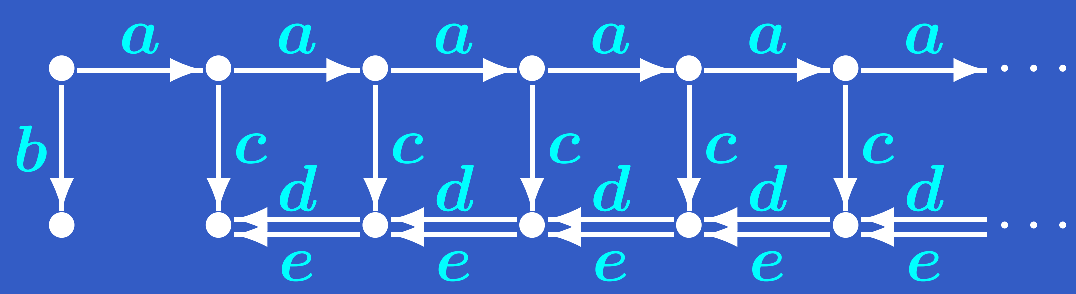

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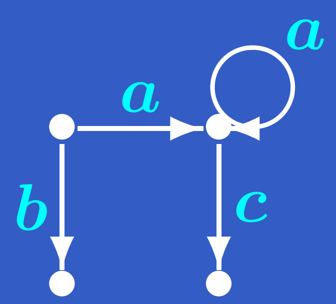

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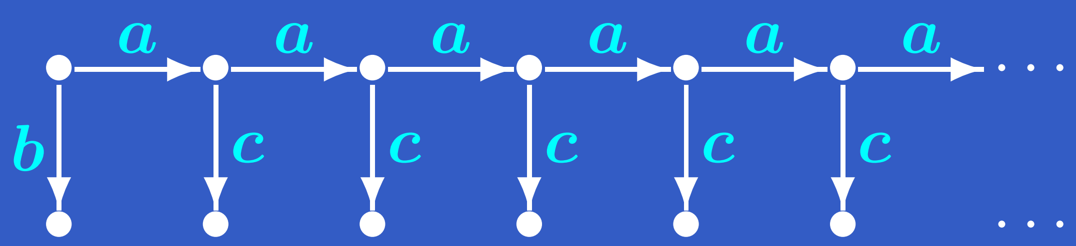

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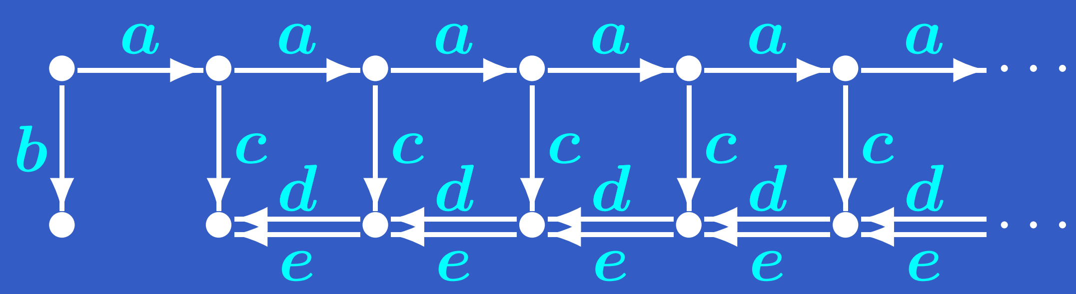

$\psi_d(x, y)$=

$$\psi_e(x, y) = \exists z \exists z' (E_a(z, z') \wedge E_c(z, y) \wedge E_c(z', x))$$

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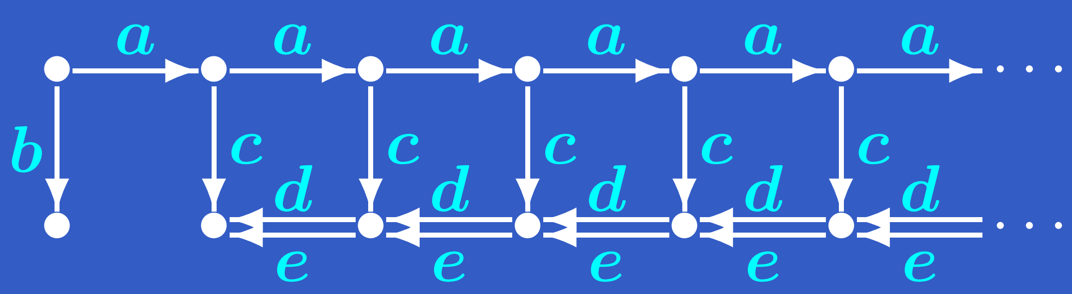

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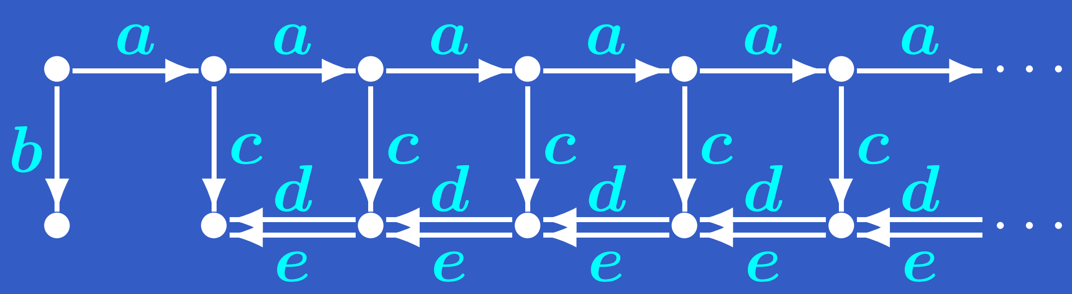
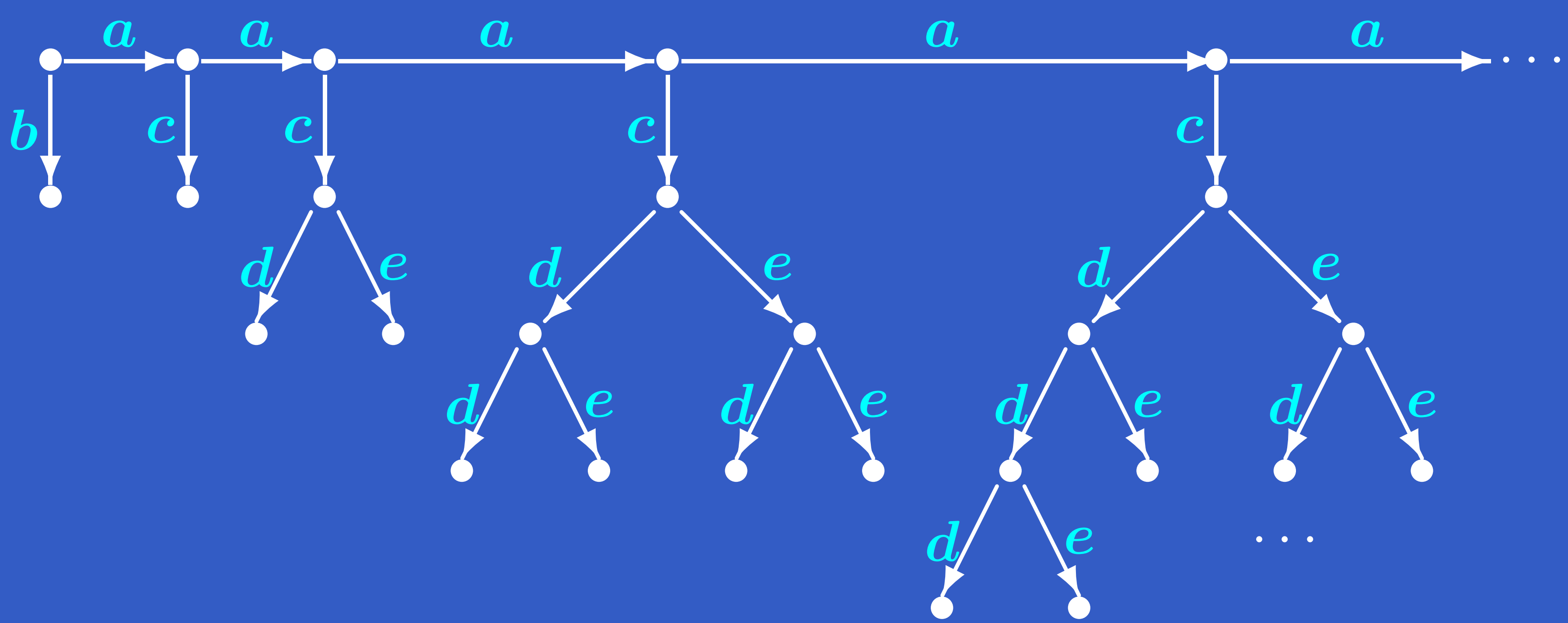


we get a tree in $\mathcal{T}_2$



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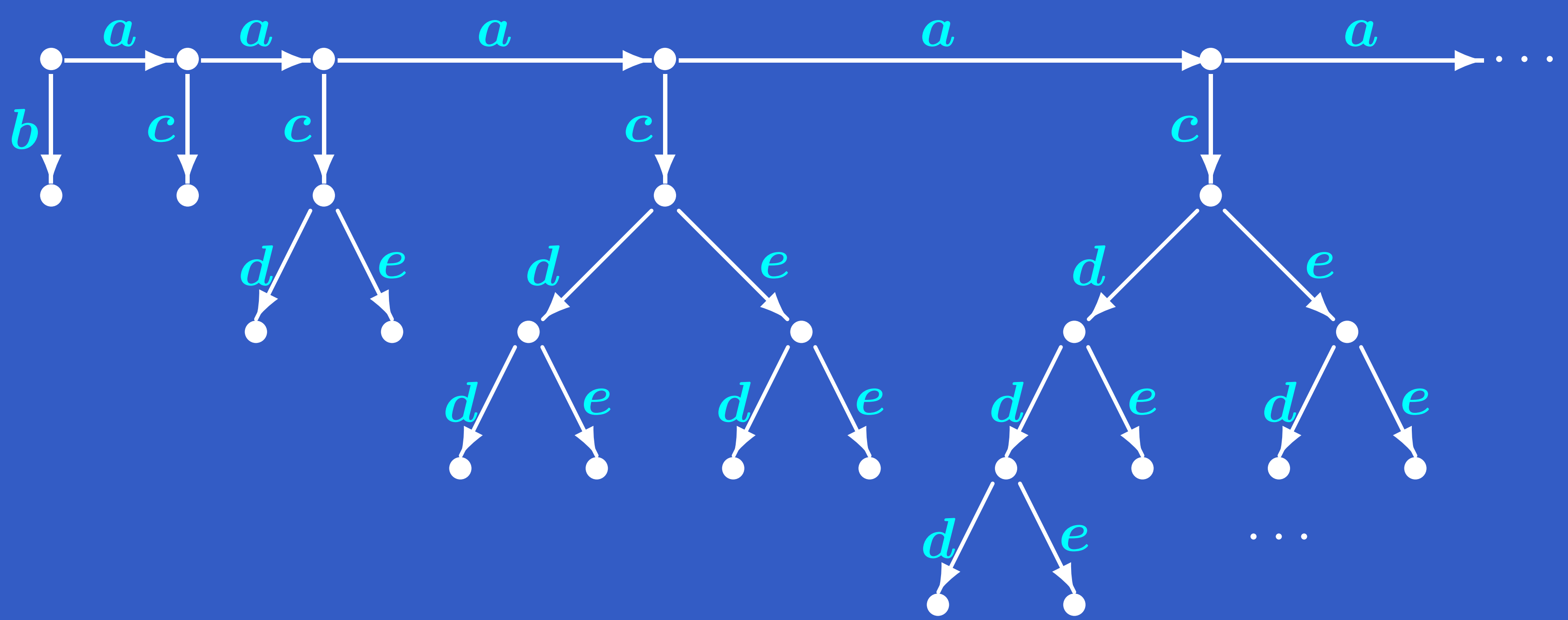

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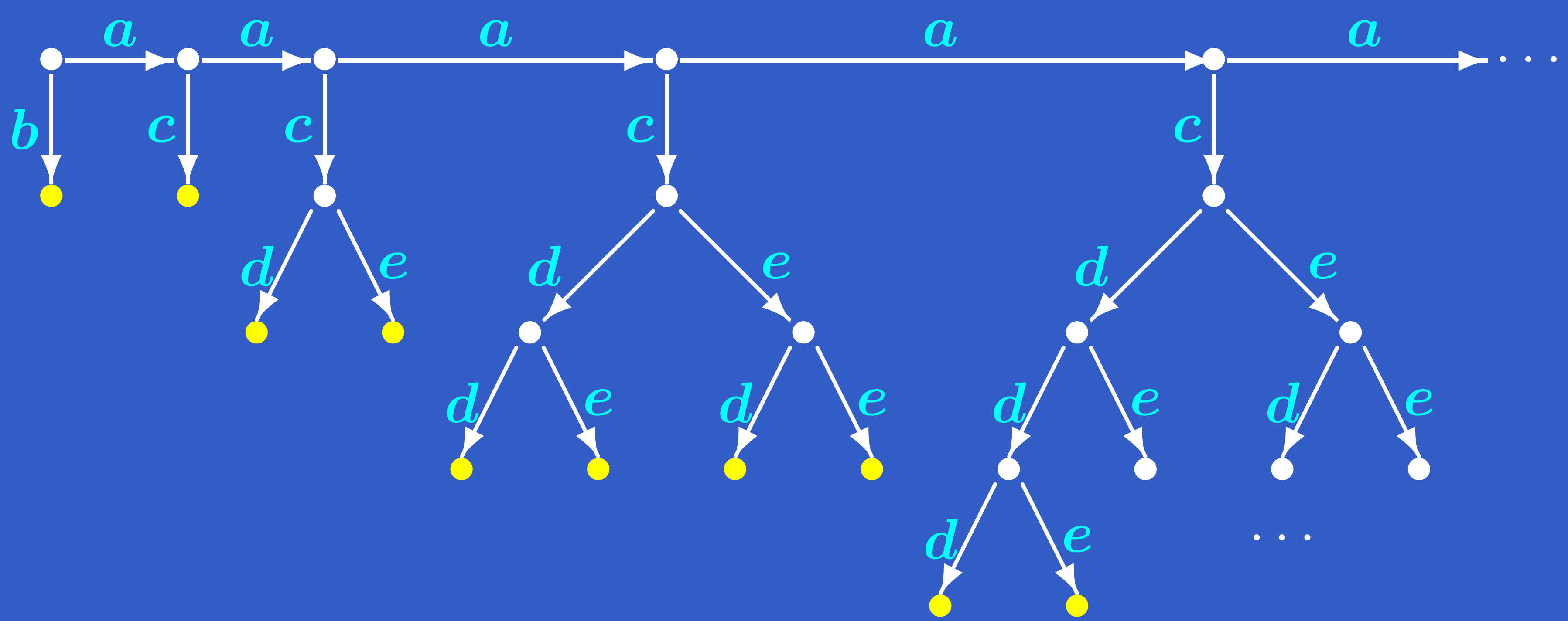

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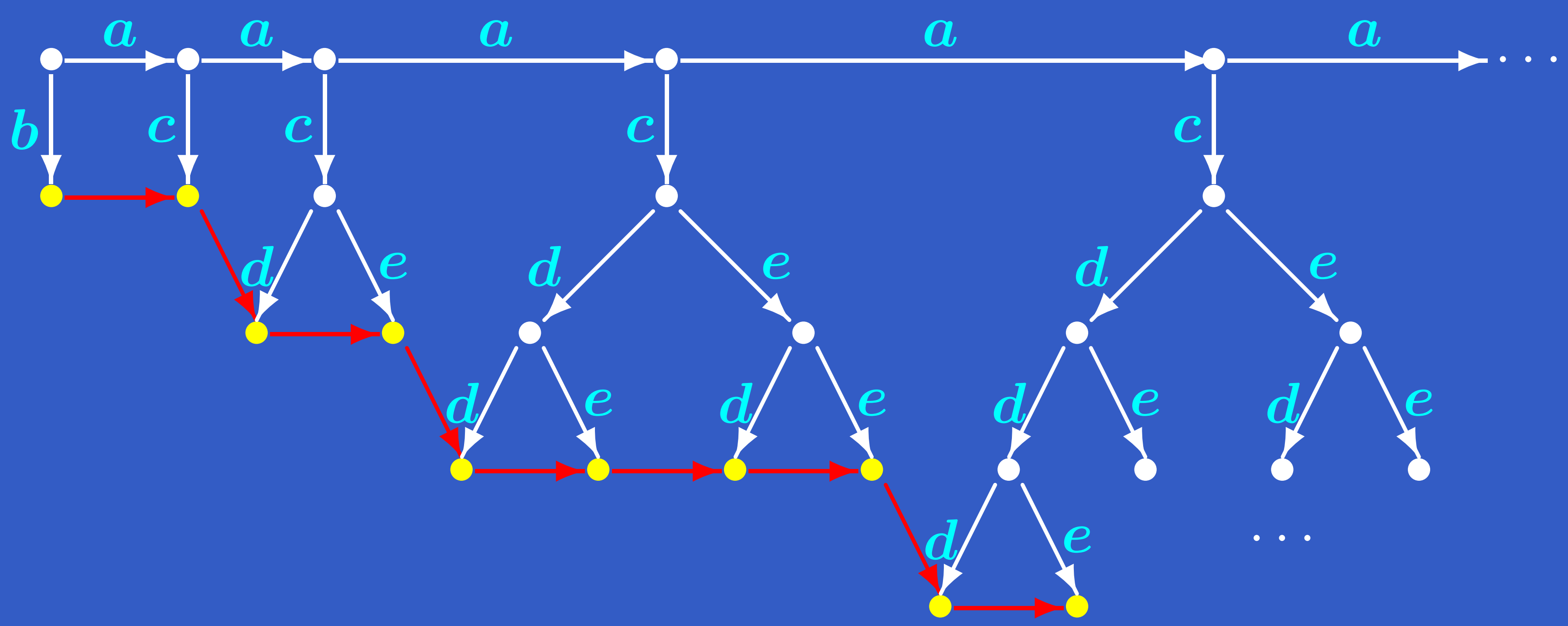

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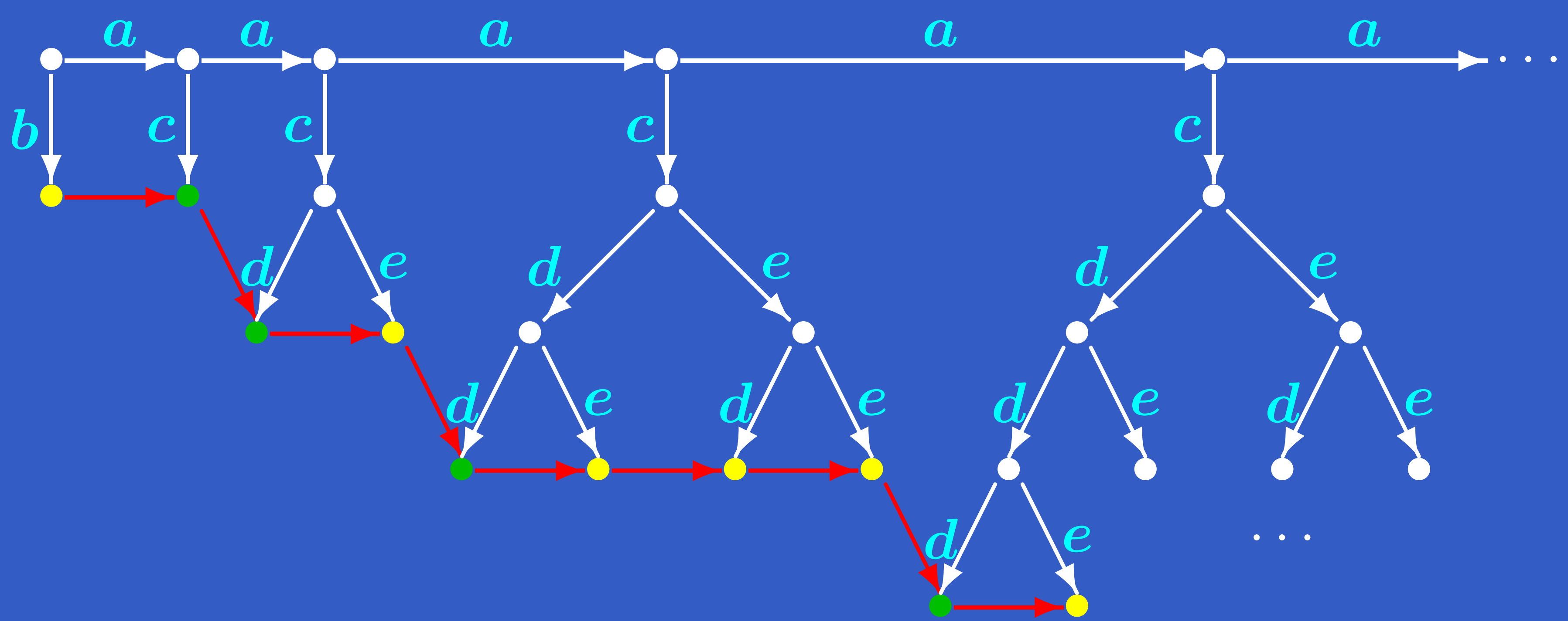

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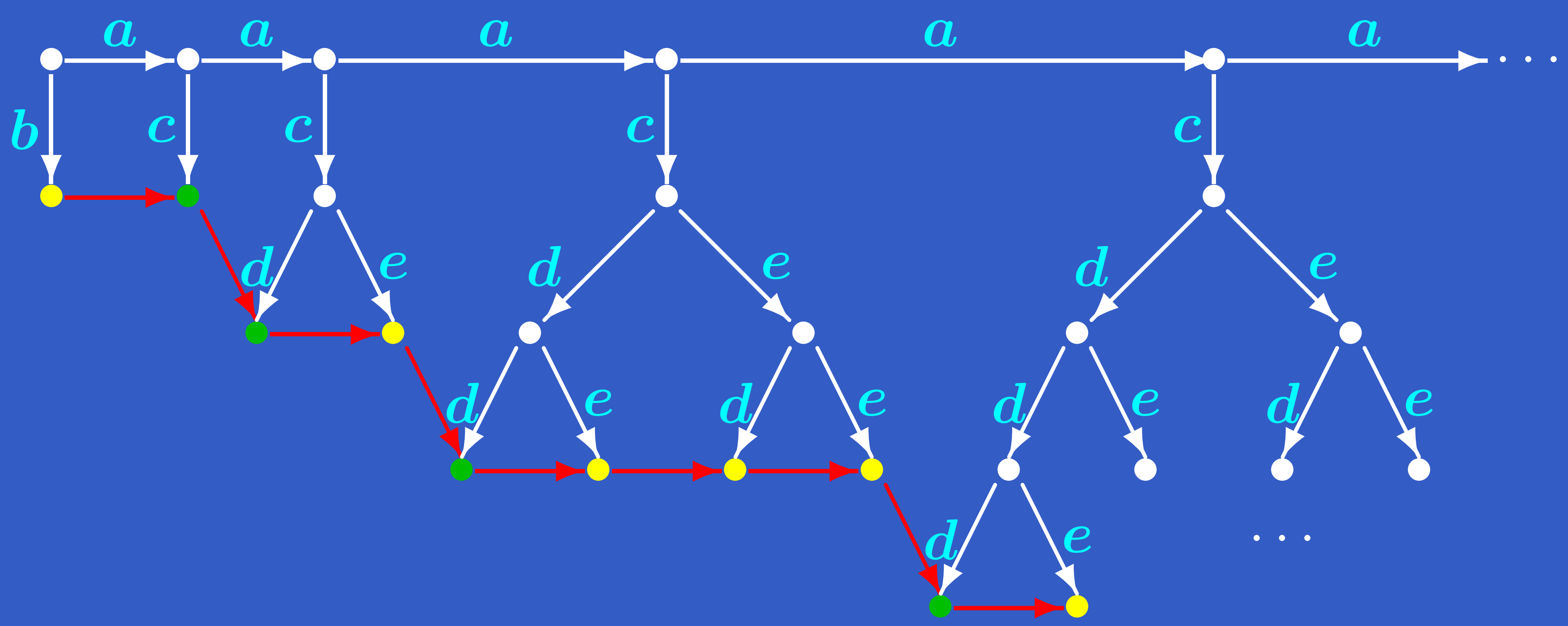

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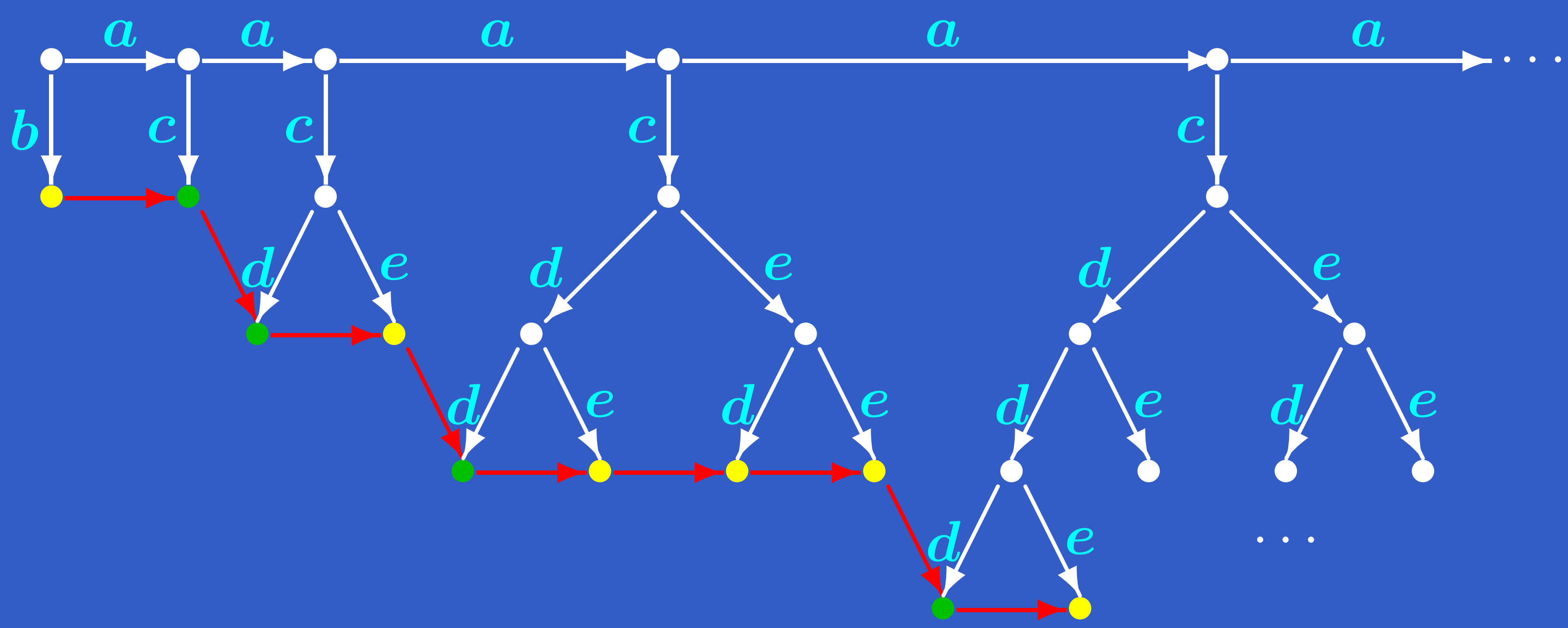

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- Original proof of decidability of MSO for $(\mathbb{N}, \text{succ}, P_2)$ by [Elgot and Rabin, 1966] was "non uniform"!

Constructing Infinite Graphs with a Decidable MSO-Theory

Wolfgang Thomas

Invited talk, MFCS 2003

The paper is available from Wolfgang Thomas's webpage.