Stacks, Queues, Priority Queues, Heaps

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Programming and Data Structures with Python

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Stack

- Stack is a last-in, first-out sequence
- `push(s,x)` — add `x` to stack `s`
- `pop(s)` — return most recently added element

s.push(x)

v ← s.pop()

isempty(s)

l.pop() in Python

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- Maintain stack as list, push and pop from the right
  - `push(s,x)` is `s.append(x)`
  - `s.pop()` — Python built-in, returns last element

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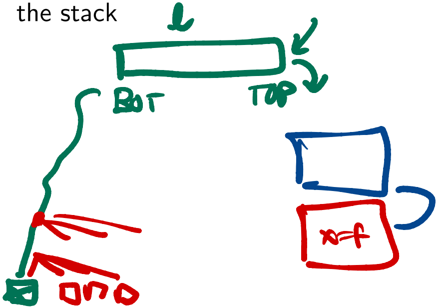
- Stacks are natural to keep track of local variables through function calls
  - Each function call pushes current frame onto a stack
  - When function exits, pop its frame off the stack



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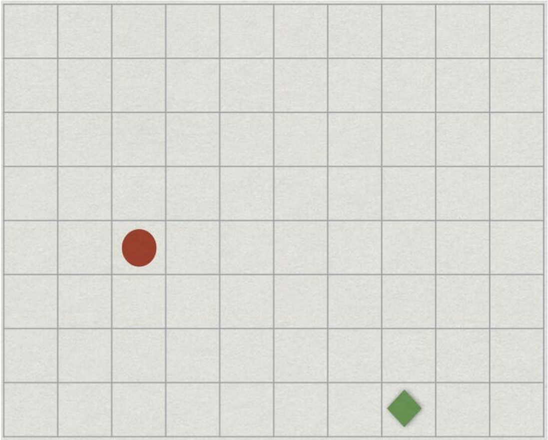
- `removeq(q)` — removes element at head of `q`
- Using Python lists, left is rear, right is front
  - `addq(q,x)` is `q.insert(0,x)`
    - `insert(j,x)`, insert `x` before position `j`
  - `removeq(q)` is `q.pop()`

Systematic exploration

- Rectangular $m \times n$ grid
- Chess knight starts at (sx, sy)
- Usual knight moves
- Can it reach a target square (tx, ty)?



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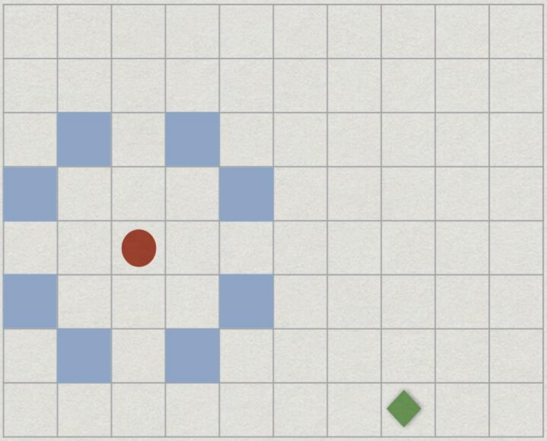

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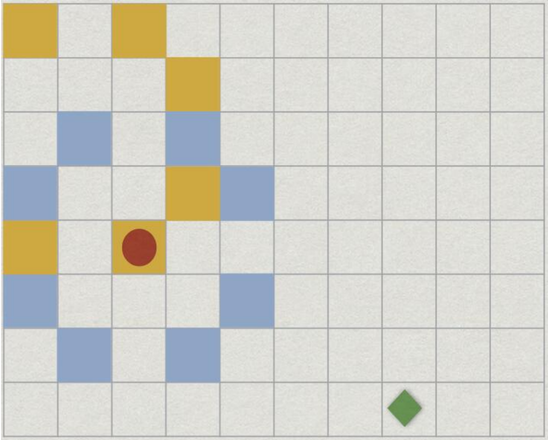

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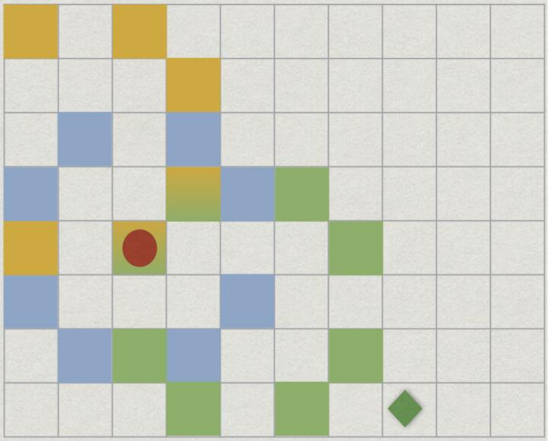

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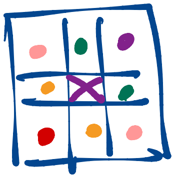

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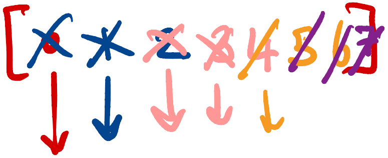

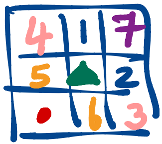

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  - Remove (ax, ay) from head of queue
  - Mark all squares reachable in one step from (ax, ay)
  - Add all newly marked squares to the queue
- When the queue is empty, we have finished

Dealing with priorities

Job scheduler

- A job scheduler maintains a list of pending jobs with their priorities

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- New jobs may join the list at any time
- How should the scheduler maintain the list of pending jobs and their priorities?

Priority queue

- Need to maintain a collection of items with priorities to optimise the following operations
- `delete_max()`
  - Identify and remove item with highest priority
  - Need not be unique
- `insert()`
  - Add a new item to the collection

Implementing priority queues with one dimensional structures

- `delete_max()`
  - Identify and remove item with highest priority
  - Need not be unique
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Implementing priority queues with one dimensional structures

- Unsorted list
  - `insert()` is $O(1)$
  - `delete_max()` is $O(n)$

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- Sorted list
  - `delete_max()` is $O(1)$
  - `insert()` is $O(n)$
- Processing n items requires $O(n^2)$

- `delete_max()`
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  - Need not be unique
- `insert()`
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Moving to two dimensions

First attempt

- Assume N processes enter/leave the queue

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- Assume N processes enter/leave the queue
- Maintain a $\sqrt{N} \times \sqrt{N}$ array

$N = 25$

3	19	23	35	58
12	17	25	43	67
10	13	~~20~~ 15		
11	16	28	49	
6	14			

Moving to two dimensions

First attempt

- Assume N processes enter/leave the queue
- Maintain a $\sqrt{N} \times \sqrt{N}$ array
- Each row is in sorted order

$N = 25$

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insert()

- Keep track of the size of each row

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5
3
4
2

insert()

- Keep track of the size of each row
- Insert into the first row that has space
  - Use size of row to determine

$N = 25$

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insert()

- Keep track of the size of each row
- Insert into the first row that has space
  - Use size of row to determine
- Insert 15
- Takes time $O(\sqrt{N})$
  - Scan size column to locate row to insert, $O(\sqrt{N})$
  - Insert into the first row with free space, $O(\sqrt{N})$

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delete_max()

- Maximum in each row is the last element

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- Identify the maximum amongst these

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delete_max()

- Maximum in each row is the last element
- Position is available through size column
- Identify the maximum amongst these
- Delete it
- Again $O(\sqrt{N})$
  - Find the maximum among last entries, $O(\sqrt{N})$
  - Delete it, $O(1)$

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Summary

- 2D $\sqrt{N} \times \sqrt{N}$ array with sorted rows
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  - Height $O(\log N)$
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- Flexible — need not fix N in advance

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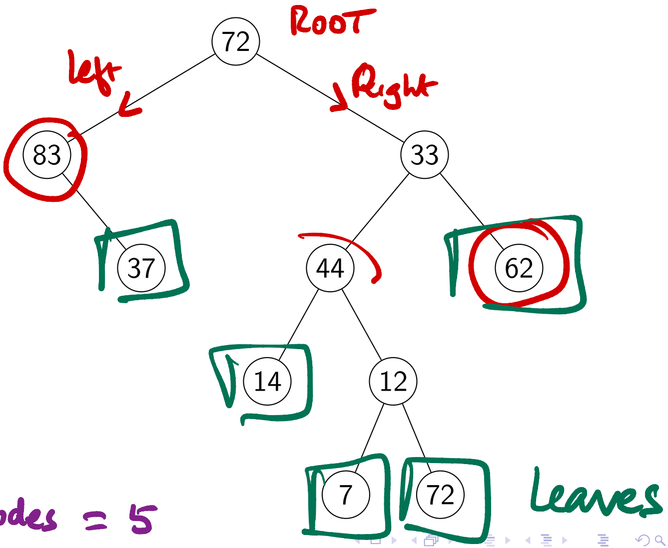
Binary trees

- Values are stored as nodes in a rooted tree
- Each node has up to two children
  - Left child and right child
  - Order is important
- Other than the root, each node has a unique parent
- Leaf node — no children
- Size — number of nodes = 10
- Height — number of levels In nodes = 5



Complete binary tree

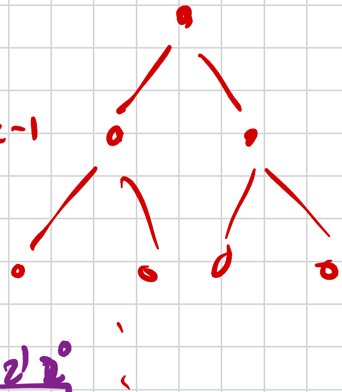
size $\approx 2^{\text{height}}$

height = log size

$1 + 2 + 4 + \ldots$

$2^0 + 2^1 + 2^2 + 2^3 \cdots + 2^{k-1}$

$= 2^k - 1$

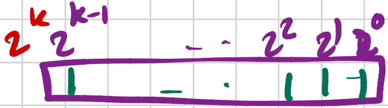


size 1 height 1

size 3 h 2

size 7 h 3

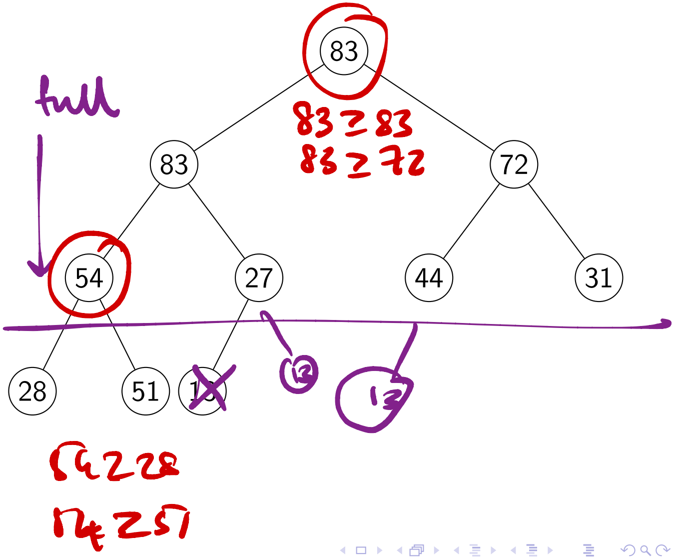
size $2^k - 1$ h k



2^k - 1 0 - - - 0

Heap

- Binary tree filled level by level, left to right   Structural constraint
- The value at each node is at least as big the values of its children   Value constraint
  - max-heap



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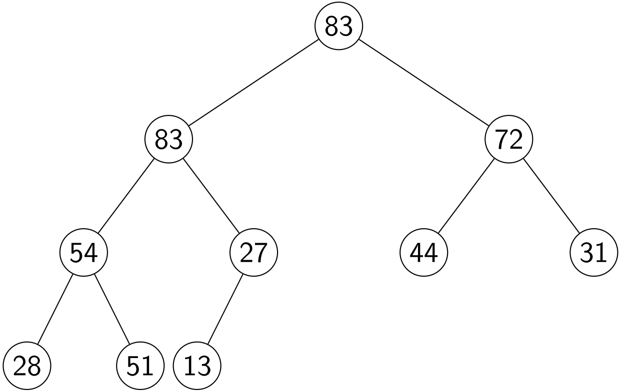

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- The value at each node is at least as big the values of its children
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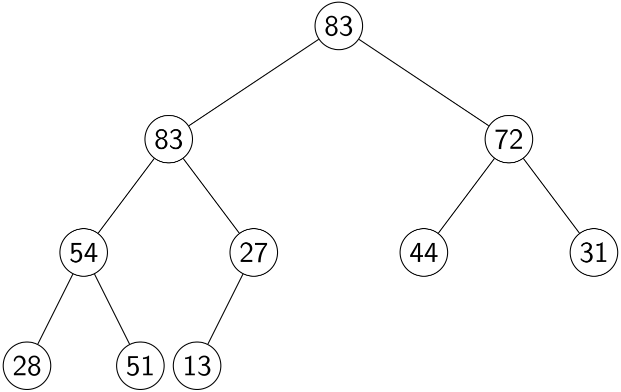
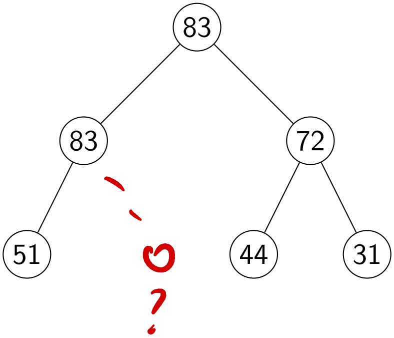
- Binary tree on the right is an example of a heap
- Root always has the largest value
  - By induction, because of the max-heap property



Non-examples

No "holes" allowed



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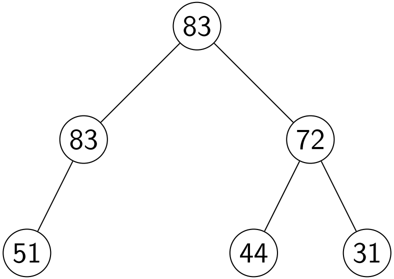
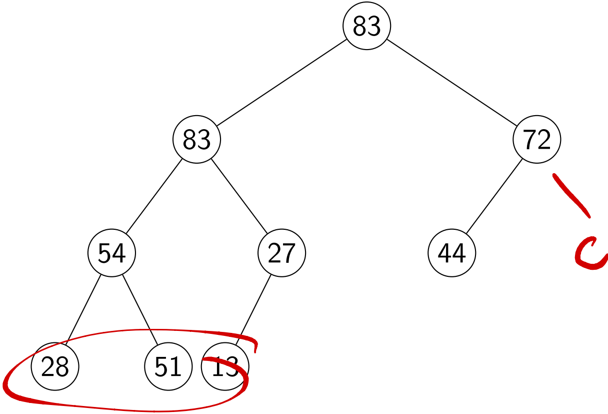


Cannot leave a level incomplete



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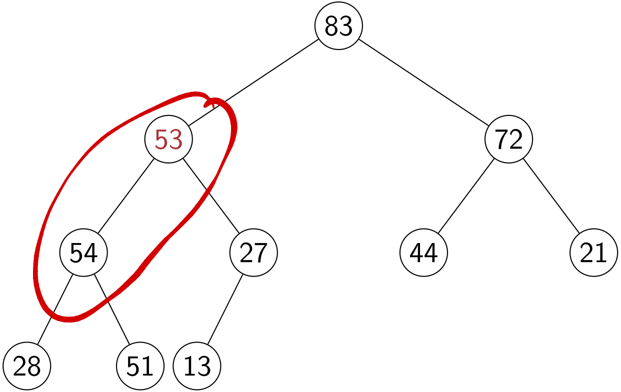
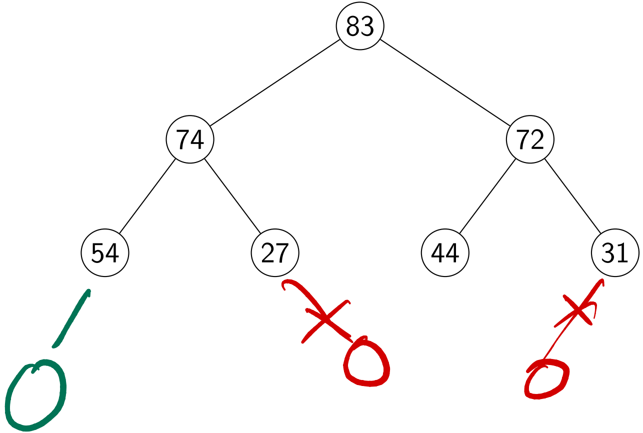
Heap property is violated



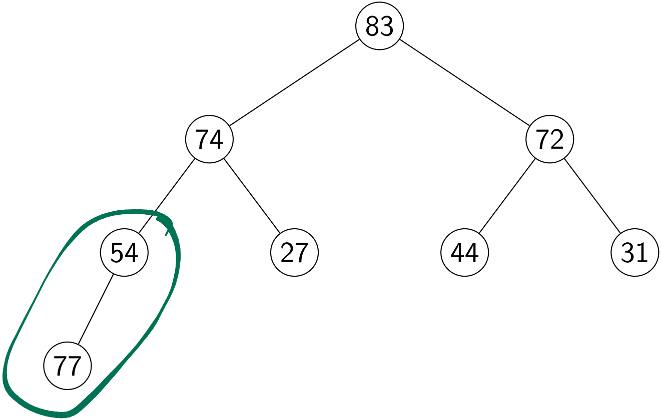
insert()

- insert(77)



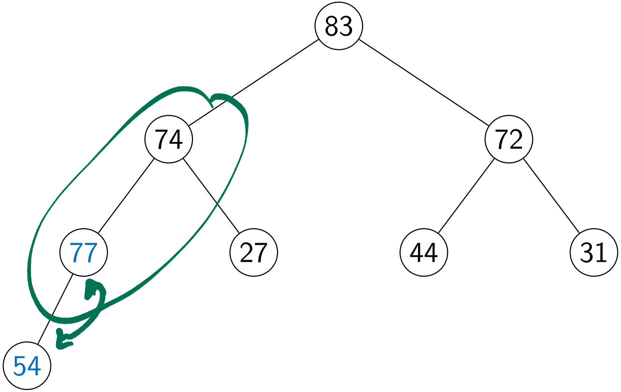
insert()

- `insert(77)`
- Add a new node at dictated by heap structure



insert()

- `insert(77)`
- Add a new node at dictated by heap structure
- Restore the heap property along path to the root



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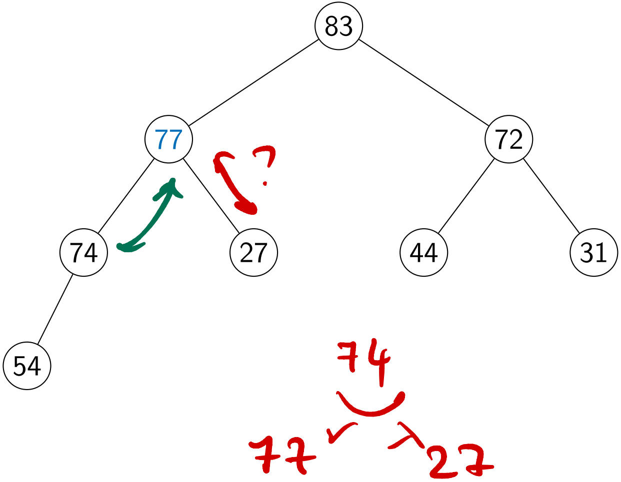

insert()

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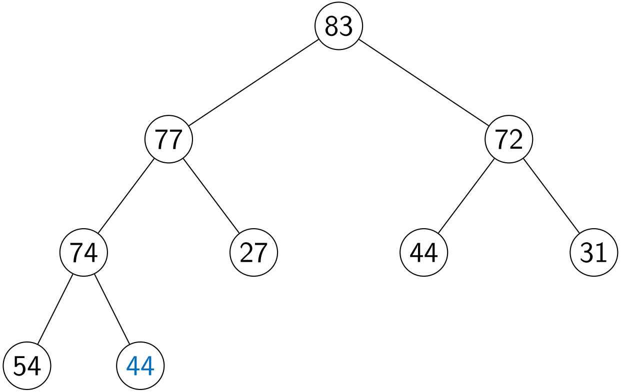
- Restore the heap property along path to the root
- insert(44)



insert()

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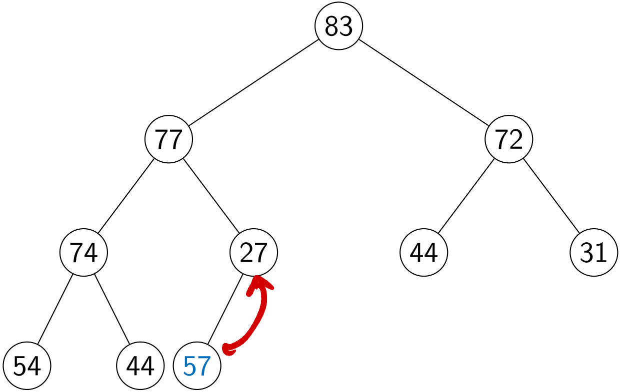
- Restore the heap property along path to the root
- `insert(44)`
- `insert(57)`



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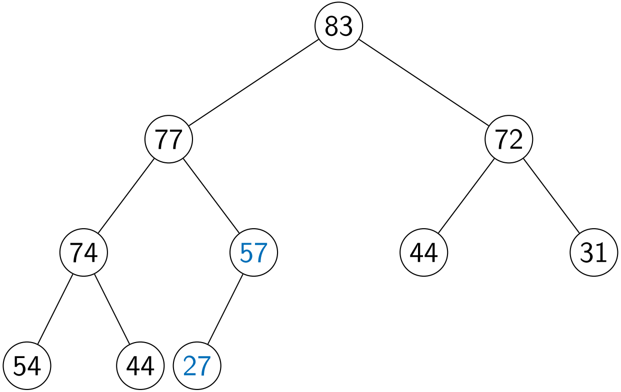
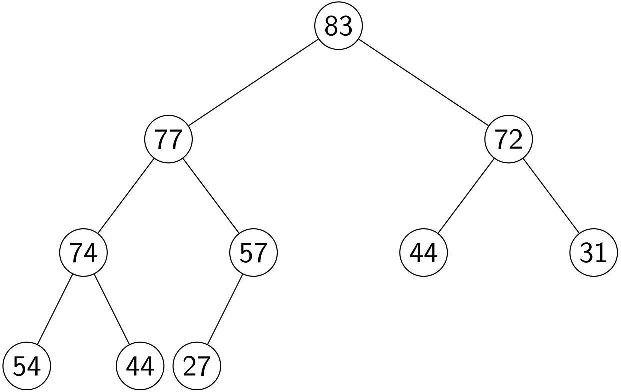


Complexity of insert()

- Need to walk up from the leaf to the root
  - Height of the tree



Complexity of `insert()`

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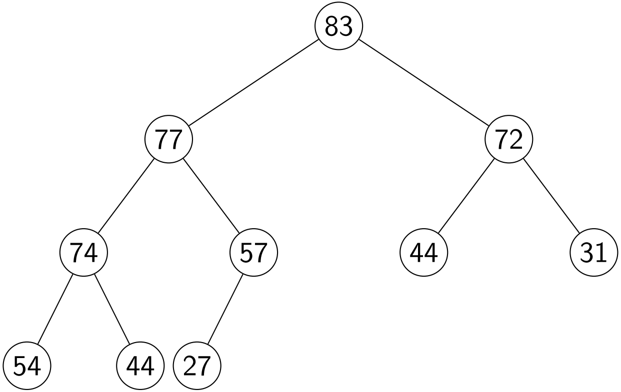
- Number of nodes at level 0 is $2^0 = 1$



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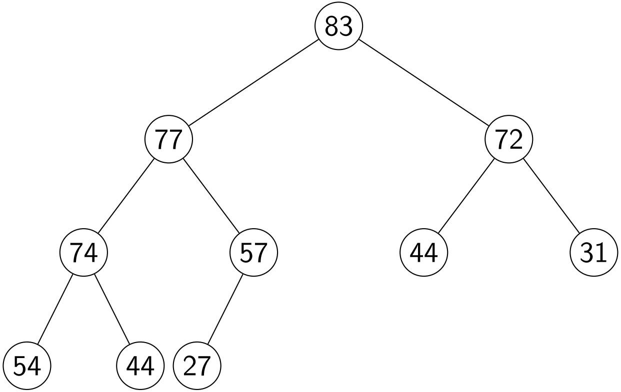

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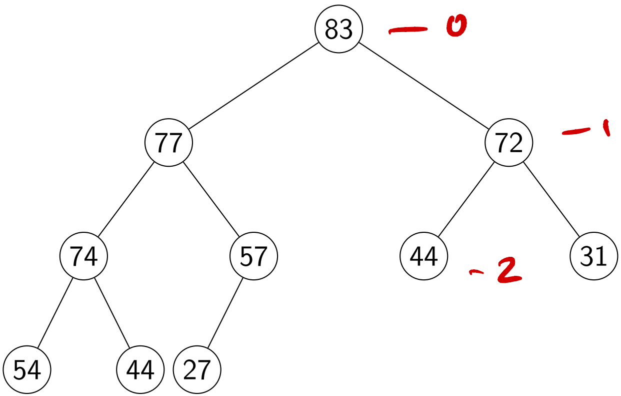

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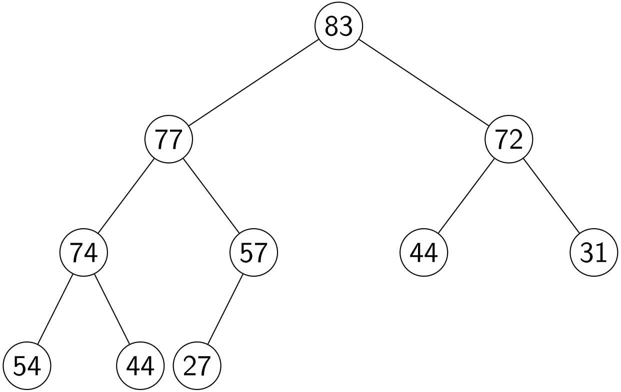

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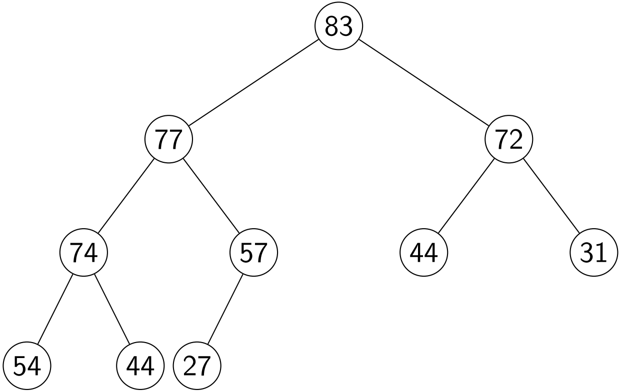
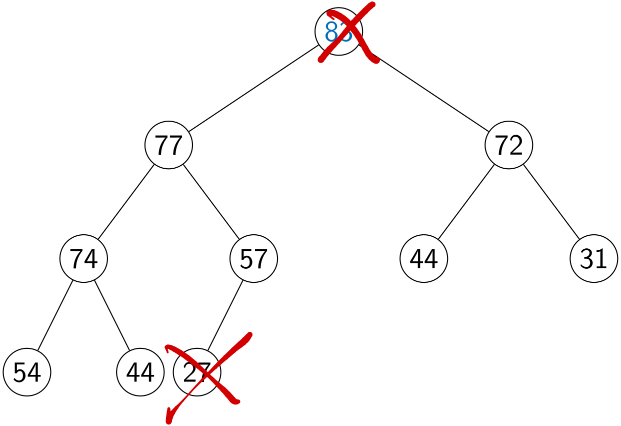


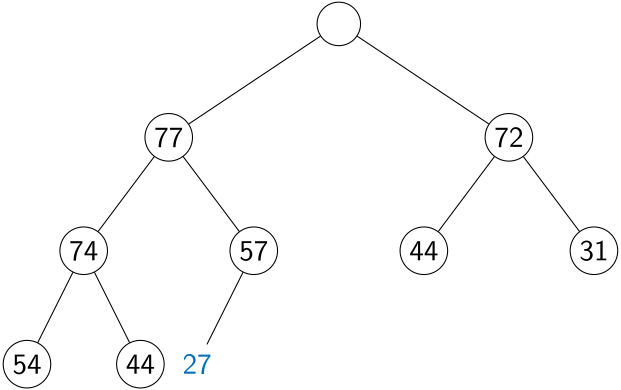
delete_max()

- Maximum value is always at the root



delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
  - Node to delete is rightmost at lowest level



delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
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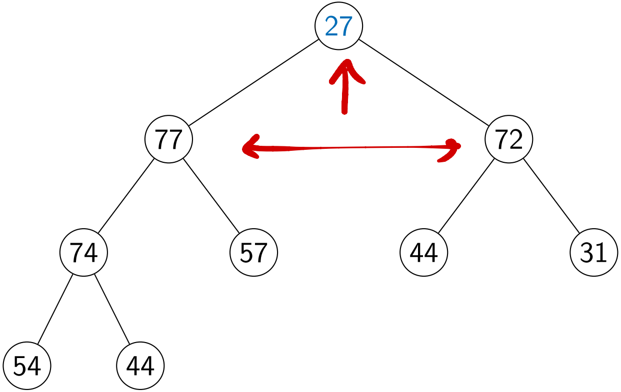
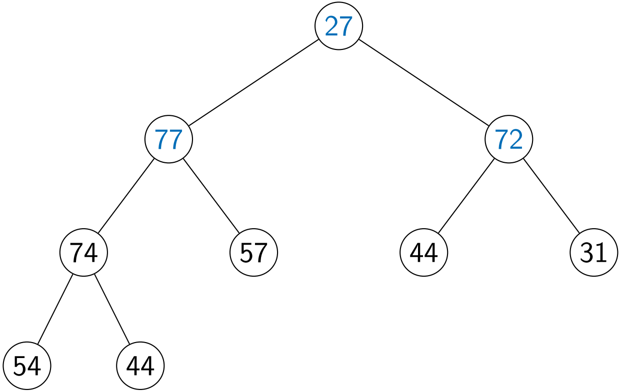
- Move "homeless" value to the root



delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
  - Node to delete is rightmost at lowest level
- Move "homeless" value to the root
- Restore the heap property downwards



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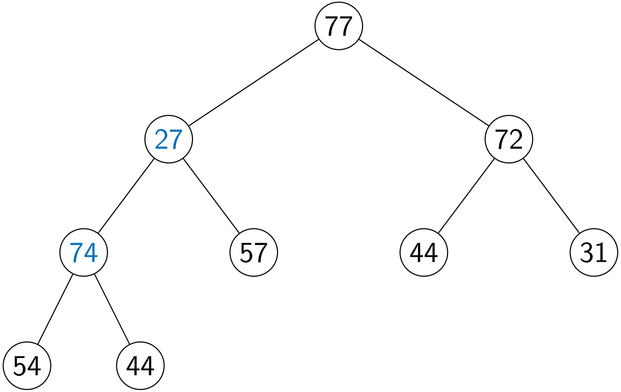
- Only need to follow a single path down
  - Again $O(\log N)$



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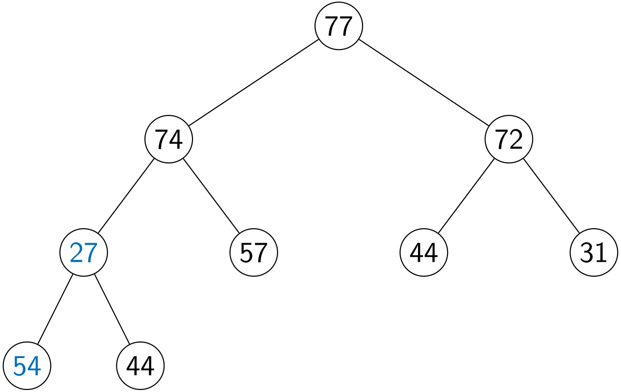

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