Quicksort

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

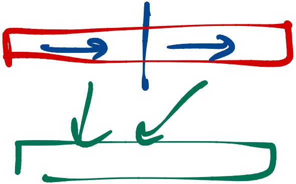
Programming and Data Structures with Python

Lecture 17, 24 Oct 2023

Shortcomings of merge sort

- Merge needs to create a new list to hold the merged elements
  - No obvious way to efficiently merge two lists in place
  - Extra storage can be costly
- Inherently recursive
  - Recursive calls and returns are expensive

DIVIDE & CONQUER



$$T(n) = 2T(n/2) + n$$

Shortcomings of merge sort

- Merge needs to create a new list to hold the merged elements
  - No obvious way to efficiently merge two lists in place
  - Extra storage can be costly
- Inherently recursive
  - Recursive calls and returns are expensive
- Merging happens because elements in the left half need to move to the right half and vice versa

  6,0,4,2 | 9,3,5,1

  - Consider an input of the form `[0,2,4,6,1,3,5,9]`

Shortcomings of merge sort

- Merge needs to create a new list to hold the merged elements
  - No obvious way to efficiently merge two lists in place
  - Extra storage can be costly
- Inherently recursive
  - Recursive calls and returns are expensive
- Merging happens because elements in the left half need to move to the right half and vice versa
  - Consider an input of the form `[0,2,4,6,1,3,5,9]`
- Can we divide the list so that everything on the left is smaller than everything on the right?
  - No need to merge!

Divide and conquer without merging

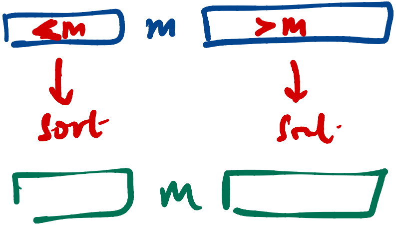
- Suppose the median of `L` is m

Divide and conquer without merging

- Suppose the median of `L` is m
- Move all values $\leq m$ to left half of `L`
  - Right half has values $> m$

Divide and conquer without merging

- Suppose the median of `L` is m
- Move all values $\leq m$ to left half of `L`
  - Right half has values $> m$
- RecursIvely sort left and right halves
  - `L` is now sorted, no merge!



Divide and conquer without merging

- Suppose the median of `L` is m
- Move all values $\leq m$ to left half of `L`
  - Right half has values $> m$
- Recurslvely sort left and right halves
  - `L` is now sorted, no merge!
- Recurrence: $T(n) = 2T(n/2) + n$
  - Rearrange in a single pass, time $O(n)$

Divide and conquer without merging

- Suppose the median of `L` is m
- Move all values $\leq m$ to left half of `L`
  - Right half has values $> m$
- RecursIvely sort left and right halves
  - `L` is now sorted, no merge!
- Recurrence: $T(n) = 2T(n/2) + n$
  - Rearrange in a single pass, time $O(n)$
- So $T(n)$ is $O(n \log n)$

Divide and conquer without merging

- Suppose the median of `L` is m
- Move all values $\leq m$ to left half of `L`
  - Right half has values $> m$
- RecursIvely sort left and right halves
  - `L` is now sorted, no merge!
- Recurrence: $T(n) = 2T(n/2) + n$
  - Rearrange in a single pass, time $O(n)$
- So $T(n)$ is $O(n \log n)$

- How do we find the median?

Divide and conquer without merging

- Suppose the median of `L` is m
- Move all values $\leq m$ to left half of `L`
  - Right half has values $> m$
- RecursIvely sort left and right halves
  - `L` is now sorted, no merge!
- Recurrence: $T(n) = 2T(n/2) + n$
  - Rearrange in a single pass, time $O(n)$
- So $T(n)$ is $O(n \log n)$

- How do we find the median?
  - Sort and pick up the middle element

Divide and conquer without merging

- Suppose the median of `L` is m
- Move all values $\leq m$ to left half of `L`
  - Right half has values $> m$
- Recurslvely sort left and right halves
  - `L` is now sorted, no merge!
- Recurrence: $T(n) = 2T(n/2) + n$
  - Rearrange in a single pass, time $O(n)$
- So $T(n)$ is $O(n \log n)$

- How do we find the median?
  - Sort and pick up the middle element
  - But our aim is to sort the list!

Divide and conquer without merging

- Suppose the median of `L` is m
- Move all values $\leq m$ to left half of `L`
  - Right half has values $> m$
- RecursIvely sort left and right halves
  - `L` is now sorted, no merge!
- Recurrence: $T(n) = 2T(n/2) + n$
  - Rearrange in a single pass, time $O(n)$
- So $T(n)$ is $O(n \log n)$

- How do we find the median?
  - Sort and pick up the middle element
  - But our aim is to sort the list!
- Instead pick some value in `L` — pivot
  - Split `L` with respect to the pivot element

Quicksort [C.A.R. Hoare]

A = Anthony = Tony

- Choose a pivot element
  - Typically the first element in the array

Quicksort [C.A.R. Hoare]

- Choose a pivot element
  - Typically the first element in the array
- Partition L into lower and upper parts with respect to the pivot

L

lower $< L[0]$

upper $> L[0]$

Quicksort [C.A.R. Hoare]

- Choose a pivot element
  - Typically the first element in the array
- Partition `L` into lower and upper parts with respect to the pivot
- Move the pivot between the lower and upper partition

Quicksort [C.A.R. Hoare]

- Choose a pivot element
  - Typically the first element in the array
- Partition `L` into lower and upper parts with respect to the pivot
- Move the pivot between the lower and upper partition
- Recursively sort the two partitions

Quicksort [C.A.R. Hoare]

- Choose a pivot element
  - Typically the first element in the array
- Partition `L` into lower and upper parts with respect to the pivot
- Move the pivot between the lower and upper partition
- Recursively sort the two partitions

High level view of quicksort

- Input list

43	32	22	78	63	57	91	13

Quicksort [C.A.R. Hoare]

- Choose a pivot element
  - Typically the first element in the array
- Partition `L` into lower and upper parts with respect to the pivot
- Move the pivot between the lower and upper partition
- Recursively sort the two partitions

High level view of quicksort

- Input list

43	32	22	78	63	57	91	13

  - Identify pivot

Quicksort [C.A.R. Hoare]

- Choose a pivot element
  - Typically the first element in the array
- Partition L into lower and upper parts with respect to the pivot
- Move the pivot between the lower and upper partition
- Recursively sort the two partitions

High level view of quicksort

- Input list

43	32	22	78	63	57	91	13

  - Identify pivot
  - Mark lower elements and upper elements

Quicksort [C.A.R. Hoare]

- Choose a pivot element
  - Typically the first element in the array
- Partition `L` into lower and upper parts with respect to the pivot
- Move the pivot between the lower and upper partition
- Recursively sort the two partitions

High level view of quicksort

- Input list

43	32	22	78	63	57	91	13

  - Identify pivot
  - Mark lower elements and upper elements
- Rearrange the elements as lower–pivot–upper

32	22	13	43	78	63	57	91

Quicksort [C.A.R. Hoare]

- Choose a pivot element
  - Typically the first element in the array
- Partition `L` into lower and upper parts with respect to the pivot
- Move the pivot between the lower and upper partition
- Recursively sort the two partitions

High level view of quicksort

- Input list

43	32	22	78	63	57	91	13

  - Identify pivot
  - Mark lower elements and upper elements
- Rearrange the elements as lower–pivot–upper

32	22	13	43	78	63	57	91

- Recursively sort the lower and upper partitions

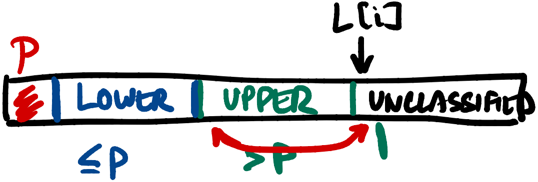
```
def quicksort(L):
    if L==0:
        return []
    pivot = L[0]   # could be L[i] for any i
    lower = [x for x in L[1:] if x <= pivot]
    upper = [x for x in L[1:] if x > pivot]

                          recursive call
    return quicksort(lower) + [pivot] + quicksort(upper)

                           new list
```

Partitioning

- Scan the list from left to right

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified



Eventually

If L[i] > P

Shift green boundary right

else

Make space in LOWER

Swap with first UPPER

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

43	32	22	78	63	57	91	13

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

43	32	22	78	63	57	91	13

- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

43	32	22	78	63	57	91	13

$\uparrow\ \uparrow$

- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

43	32	22	78	63	57	91	13

- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

43	32	22	78	63	57	91	13

- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

43	32	22	78	63	57	91	13
			$\uparrow$		$\uparrow$		

- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

43	32	22	78	63	57	91	13

- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments

Partitioning

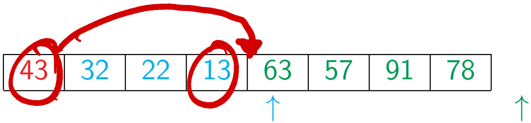
- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

43	32	22	78	63	57	91	13

- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.



- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments

Partitioning

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Examine the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

13	32	22	43	63	57	91	78

- Pivot is always the first element
- Maintain two indices to mark the end of the Lower and Upper segments
- After partitioning, exchange the pivot with the last element of the Lower segment

Quicksort code

- Scan the list from left to right
- Four segments: Pivot, Lower, Upper, Unclassified
- Classify the first unclassified element
  - If it is larger than the pivot, extend Upper to include this element
  - If it is less than or equal to the pivot, exchange with the first element in Upper. This extends Lower and shifts Upper by one position.

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):   ← l to r-1
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Summary

- Quicksort uses divide and conquer, like merge sort

Summary

- Quicksort uses divide and conquer, like merge sort
- By partitioning the list carefully, we avoid a merge step
  - This allows an in place sort

Summary

- Quicksort uses divide and conquer, like merge sort
- By partitioning the list carefully, we avoid a merge step
  - This allows an in place sort
- We can also provide an iterative implementation to avoid the cost of recursive calls

Summary

- Quicksort uses divide and conquer, like merge sort
- By partitioning the list carefully, we avoid a merge step
  - This allows an in place sort
- We can also provide an iterative implementation to avoid the cost of recursive calls
- The partitioning strategy we described is not the only one used in the literature
  - Can build the lower and upper segments from opposite ends and meet in the middle

Summary

- Quicksort uses divide and conquer, like merge sort
- By partitioning the list carefully, we avoid a merge step
  - This allows an in place sort
- We can also provide an iterative implementation to avoid the cost of recursive calls
- The partitioning strategy we described is not the only one used in the literature
  - Can build the lower and upper segments from opposite ends and meet in the middle
- Need to analyse the complexity of quick sort

Analysis

- Partitioning with respect to the pivot takes time $O(n)$

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Analysis

- Partitioning with respect to the pivot takes time $O(n)$
- If the pivot is the median
  - $T(n) = 2T(n/2) + n$
  - $T(n)$ is $O(n \log n)$

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Analysis

- Partitioning with respect to the pivot takes time $O(n)$
- If the pivot is the median
  - $T(n) = 2T(n/2) + n$
  - $T(n)$ is $O(n \log n)$
- Worst case? Pivot is maximum or minimum
  - Partitions are of size 0, $n-1$
  - $T(n) = T(n-1) + n$
  - $T(n) = n + (n-1) + \cdots + 1$
  - $T(n)$ is $O(n^2)$

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Analysis

- Partitioning with respect to the pivot takes time $O(n)$
- If the pivot is the median
  - $T(n) = 2T(n/2) + n$
  - $T(n)$ is $O(n \log n)$
- Worst case? Pivot is maximum or minimum
  - Partitions are of size 0, $n-1$
  - $T(n) = T(n-1) + n$
  - $T(n) = n + (n-1) + \cdots + 1$
  - $T(n)$ is $O(n^2)$
- Already sorted array: worst case!

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Analysis . . .

- However, average case is $O(n \log n)$

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Analysis . . .

- However, average case is $O(n \log n)$
- Sorting is a rare situation where we can compute this
  - Values don't matter, only relative order is important
  - Analyze behaviour over permutations of $\{1, 2, \ldots, n\}$
  - Each input permutation equally likely

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Analysis . . .

- However, average case is $O(n \log n)$
- Sorting is a rare situation where we can compute this
  - Values don't matter, only relative order is important
  - Analyze behaviour over permutations of $\{1, 2, \ldots, n\}$
  - Each input permutation equally likely
- Expected running time is $O(n \log n)$

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Randomization

- Any fixed choice of pivot allows us to construct worst case input

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Randomization

- Any fixed choice of pivot allows us to construct worst case input
- Instead, choose pivot position randomly at each step

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Randomization

- Any fixed choice of pivot allows us to construct worst case input
- Instead, choose pivot position randomly at each step
- Expected running time is again $O(n \log n)$

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Iterative quicksort

- Recursive calls work on disjoint segments
  - No recombination of results is required

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Iterative quicksort

- Recursive calls work on disjoint segments
  - No recombination of results is required
- Can explicitly keep track of left and right endpoints of each segment to be sorted

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Quicksort in practice

- In practice, quicksort is very fast

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Quicksort in practice

- In practice, quicksort is very fast
- Very often the default algorithm used for in-built sort functions
  - Sorting a column in a spreadsheet
  - Library sort function in a programming language

```
def quicksort(L,l,r):  # Sort L[l:r]
  if (r - l <= 1):
    return(L)
  (pivot,lower,upper) = (L[l],l+1,l+1)
  for i in range(l+1,r):
    if L[i] > pivot:  # Extend upper segment
      upper = upper+1
    else: # Exchange L[i] with start of upper segment
      (L[i], L[lower]) = (L[lower], L[i])
      # Shift both segments
      (lower,upper) = (lower+1,upper+1)
  # Move pivot between lower and upper
  (L[l],L[lower-1]) = (L[lower-1],L[l])
  lower = lower-1
  # Recursive calls
  quicksort(L,l,lower)
  quicksort(L,lower+1,upper)
  return(L)
```

Summary

- The worst case complexity of quicksort is $O(n^2)$

Summary

- The worst case complexity of quicksort is $O(n^2)$
- However, the average case is $O(n \log n)$

Summary

- The worst case complexity of quicksort is $O(n^2)$
- However, the average case is $O(n \log n)$
- Randomly choosing the pivot is a good strategy to beat worst case inputs

Summary

- The worst case complexity of quicksort is $O(n^2)$
- However, the average case is $O(n \log n)$
- Randomly choosing the pivot is a good strategy to beat worst case inputs
- Quicksort works in-place and can be implemented iteratively

Summary

- The worst case complexity of quicksort is $O(n^2)$
- However, the average case is $O(n \log n)$
- Randomly choosing the pivot is a good strategy to beat worst case inputs
- Quicksort works in-place and can be implemented iteratively
- Very fast in practice, and often used for built-in sorting functions
  - Good example of a situation when the worst case upper bound is pessimistic