Recursive Insertion Sort

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming and Data Structures with Python

Lecture 16, 19 Oct 2023

Insertion sort

- You are the TA for a course
  - Instructor has a pile of evaluated exam papers
  - Papers in random order of marks
  - Your task is to arrange the papers in descending order of marks

Strategy 2

- Move the first paper to a new pile
- Second paper
  - Lower marks than first paper? Place below first paper in new pile
  - Higher marks than first paper? Place above first paper in new pile
- Third paper
  - Insert into correct position with respect to first two
- Do this for the remaining papers
  - Insert each one into correct position in the second pile

Insertion sort

- Start building a new sorted list
- Pick next element and insert it into the sorted list
- An iterative formulation
  - Assume `L[:i]` is sorted
  - Insert `L[i]` in `L[:i]`

```
def InsertionSort(L):
  n = len(L)
  if n < 1:
    return(L)
  for i in range(n):
    # Assume L[:i] is sorted
    # Move L[i] to correct position in L
    j = i
    while(j > 0 and L[j] < L[j-1]):
      (L[j],L[j-1]) = (L[j-1],L[j])
      j = j-1
    # Now L[:i+1] is sorted
  return(L)
```

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- A recursive formulation
  - Inductively sort `L[:i]`
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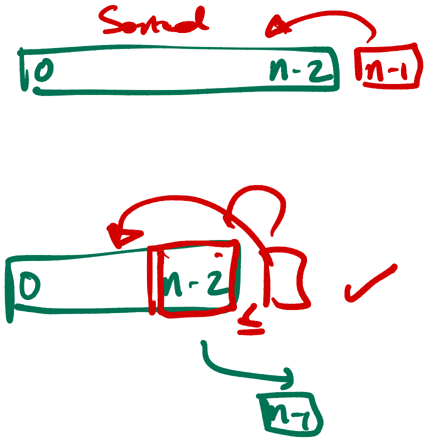

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```
def Insert(L,v):
    n = len(L)
    if n == 0:
      return([v])
    if v >= L[-1]:
      return(L+[v])
    else:
      return(Insert(L[:-1],v)+L[-1:])

def ISort(L):
    n = len(L)
    if n < 1:
       return(L)
    L = Insert(ISort(L[:-1]),L[-1])
    return(L)
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Analysis of recursive insertion sort

- For input of size n, let
  - $TI(n)$ be the time taken by `Insert`
  - $TS(n)$ be the time taken by `ISort`

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- Unwind to get $1 + 2 + \cdots + n - 1$

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Merge Sort

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Beating the $O(n^2)$ barrier

- Both selection sort and insertion sort take time $O(n^2)$
- This is infeasible for $n > 10000$
- How can we bring the complexity below $O(n^2)$?

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Strategy 3

- Divide the list into two halves
- Separately sort the left and right half
- Combine the two sorted halves to get a fully sorted list

Combining two sorted lists

- Combine two sorted lists `A` and `B` into a single sorted list `C`

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- Merging `A` and `B`

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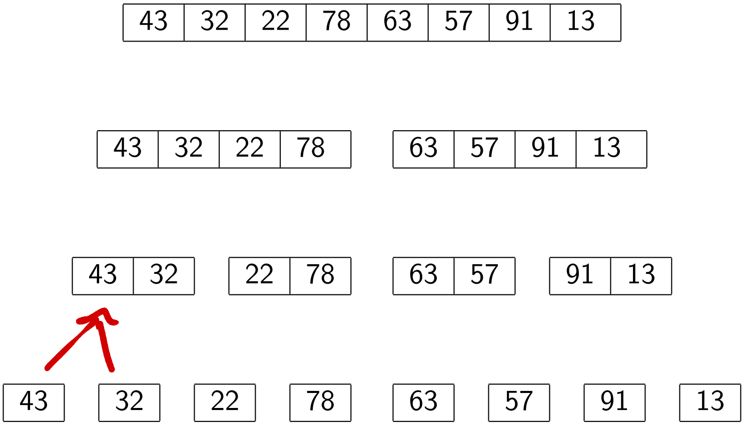

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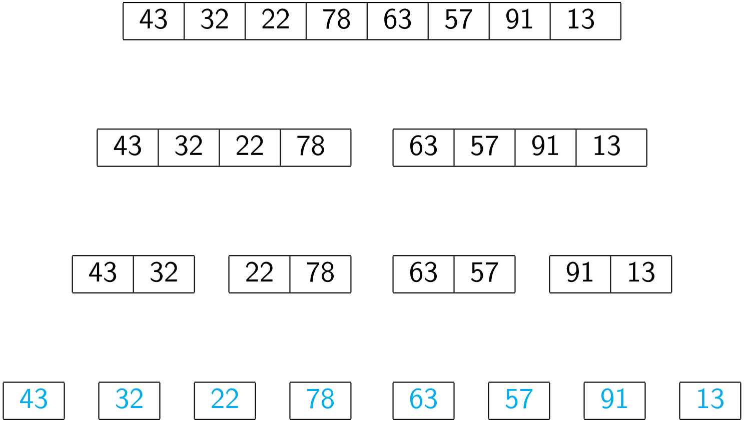

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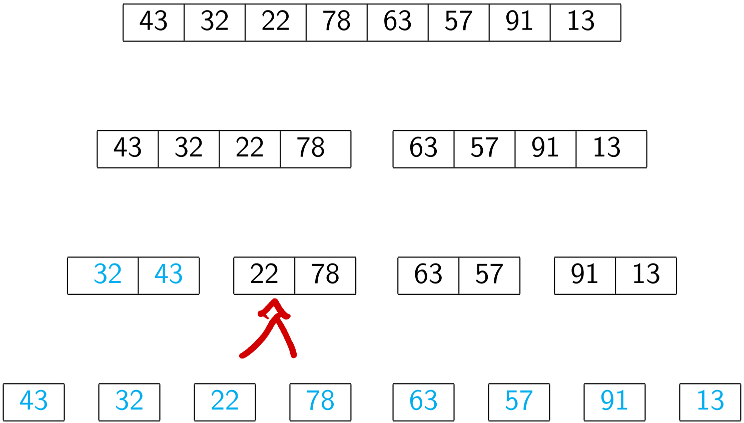

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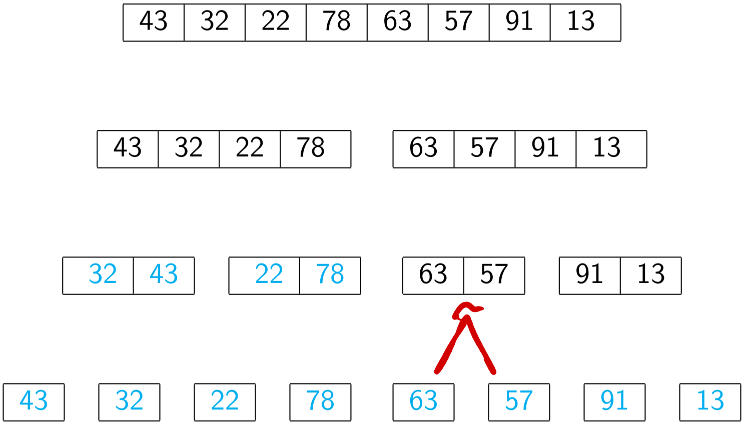

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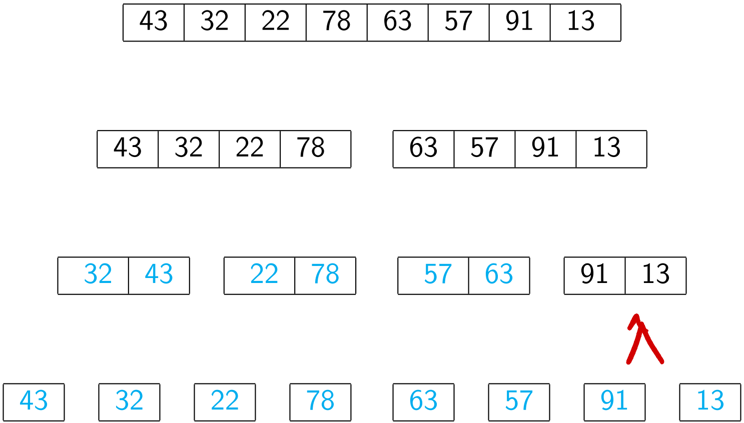

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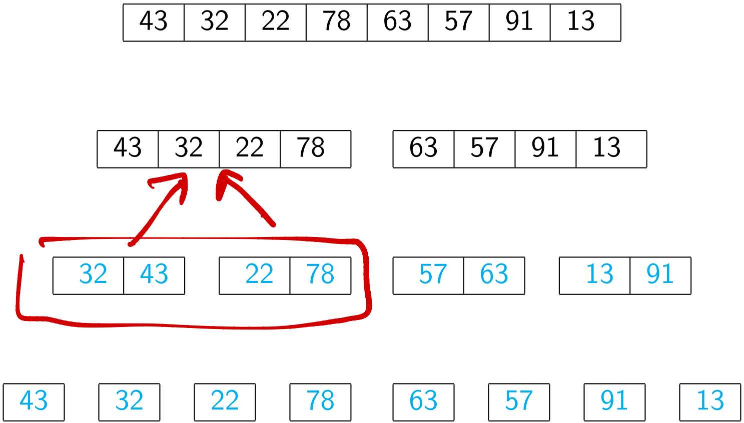

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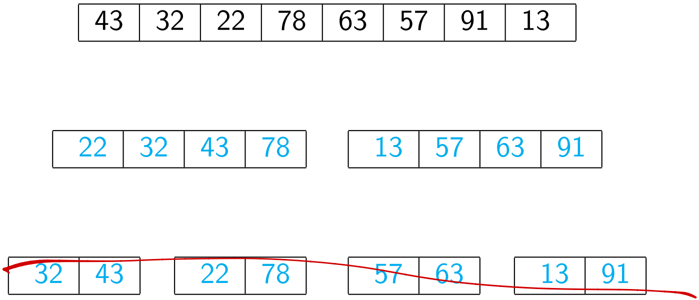

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Divide and Conquer

- Break up the problem into disjoint parts
- Solve each part separately
- Combine the solutions efficiently

Merging sorted lists

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  - If `A` is empty, copy `B` into `C`

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  - Repeat till all elements of `A` and `B` have been moved

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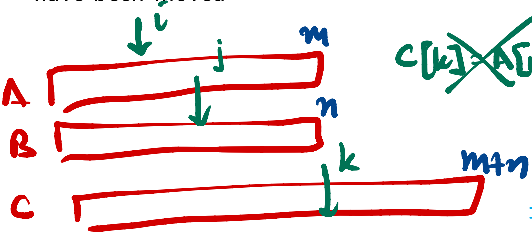

~~C[k] = A[i]~~

```
def merge(A,B):
  (m,n) = (len(A),len(B))
  (C,i,j,k) = ([],0,0,0)
  while k < m+n:
    if i == m:
      C.extend(B[j:])
      k = k + (n-j)
    elif j == n:
      C.extend(A[i:])
      k = k + (m-i)
    elif A[i] < B[j]:
      C.append(A[i])
      (i,k) = (i+1,k+1)
    else:
      C.append(B[j])
      (j,k) = (j+1,k+1)
  return(C)
```

len(C) < m+n

Boundary

Real merge

Merge sort

- To sort `A` into `B`, both of length n

Merge sort

- To sort `A` into `B`, both of length n
- If $n \leq 1$, nothing to be done

Merge sort

- To sort `A` into `B`, both of length n
- If $n \leq 1$, nothing to be done
- Otherwise

not n=0
else you will keep
splitting singleton list

Merge sort

- To sort `A` into `B`, both of length n
- If $n \leq 1$, nothing to be done
- Otherwise
  - Sort `A[:n//2]` into `L`

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- If $n \leq 1$, nothing to be done
- Otherwise
  - Sort `A[:n//2]` into `L`
  - Sort `A[n//2:]` into `R`
  - Merge `L` and `R` into `B`

Merge sort

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  - Merge `L` and `R` into `B`

```
def mergesort(A):
  n = len(A)

  if n <= 1:
     return(A)

  L = mergesort(A[:n//2])
  R = mergesort(A[n//2:])

  B = merge(L,R)

  return(B)
```

Analysing merge

- Merge A of length m, B of length n

merge (A, B)

if A == [] :
    return B
elif B == [] :
    return A
elif A[0] < B[0] :
    return ([A[0]] + merge(A[1:], B))
--

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      C.append(A[i])
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    else:
      C.append(B[j])
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  return(C)
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Analysing merge

- Merge `A` of length m, `B` of length n
- Output list `C` has length $m + n$

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  - For simplicity, assume $n = 2^k$ for some k

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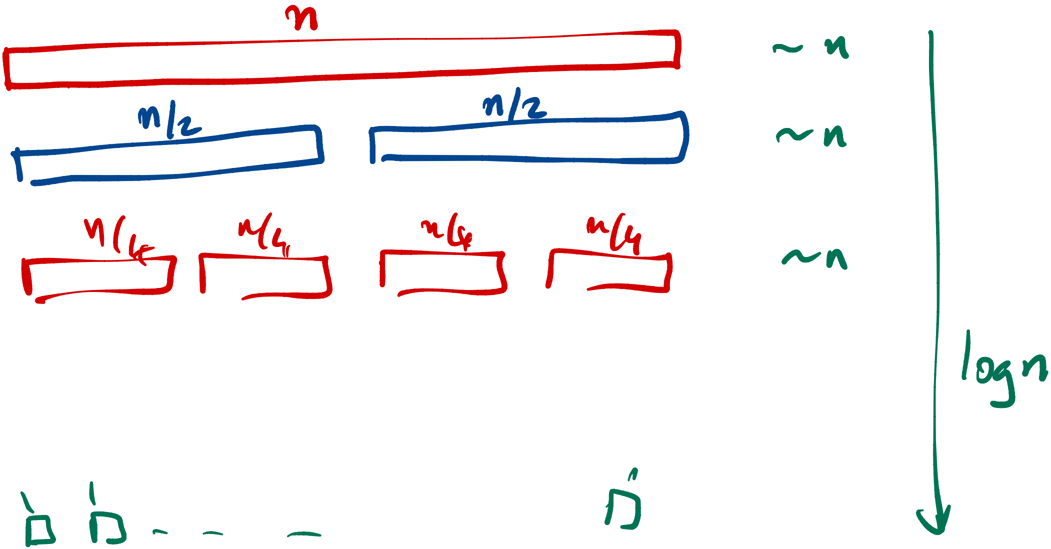

n
n/2
n/2
n/4
n/4
n/4
n/4
~n
~n
~n
log n

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l1.extend(l2)
↓
l1 + l2

l1.append(v)
l1 + [v]