Dynamic Programming

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Programming and Data Structures with Python

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Memoizing recursive implementations

```
def fib(n):
  if n in fibtable.keys():
    return(fibtable[n])
  if n <= 1:
    value = n
  else:
    value = fib(n-1) + fib(n-2)
  fibtable[n] = value
  return(value)
```

In general

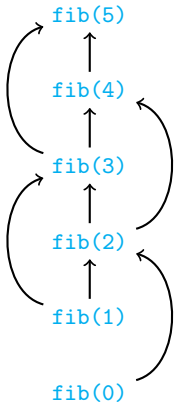
```
def f(x,y,z):
  if (x,y,z) in ftable.keys():
    return(ftable[(x,y,z)])
  recursively compute value
    from subproblems
  ftable[(x,y,z)] = value
  return(value)
```

Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic
- Solve subproblems in appropriate order
  - Start with base cases — no dependencies
  - Evaluate a value after all its dependencies are available
  - Fill table iteratively
  - Never need to make a recursive call

n	0	1	2	3	4	5
fib(n)	0	1	1	2	3	5

Evaluating `fib(5)`



Some examples

- Number of grid paths from $(0, 0)$ to (m, n)
  - $P(i, j) = P(i-1, j) + P(i, j-1)$
  - Fill in a grid of size mn
  - Complexity $O(mn)$

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  - Complexity $O(mn)$
- Longest common subword of $a_0 a_1 \ldots a_{m-1}$ and $b_0 b_1 \ldots b_{n-1}$
  - $LCW(i, j) = 0$ or $LCW(i, j) = 1 + LCW(i+1, j+1)$
  - Fill in a grid of size $(m+1)(n+1)$
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- Longest common subsequence
  - $LCS(i,j) = \max[LCS(i+1,j), LCS(i,j+1]$ or $LCS(i,j) = 1 + LCS(i+1,j+1)$
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- Multiply matrices A, B
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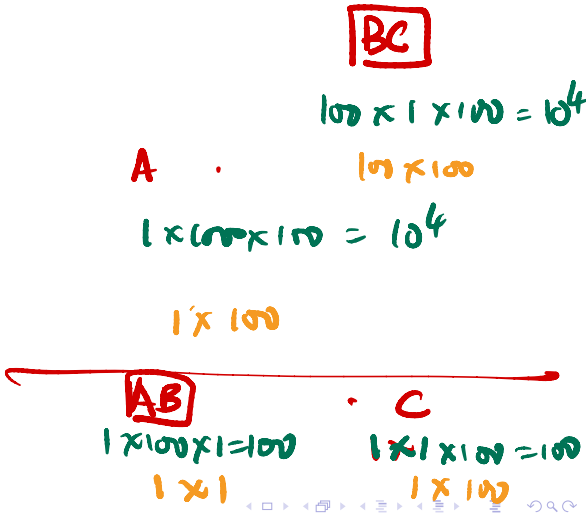

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  - Dimensions match: $r_j = c_{j-1}$, $0 < j < n$
  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed
- Find an optimal order to compute the product
  - Multiply two matrices at a time
  - Bracket the expression optimally

Inductive structure

- Final step combines two subproducts
  $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$
  for some $0 < k < n$

$1 \leq k \leq n-1$

$k = n-1$

$(M_0 - - \quad M_{n-2}) \cdot M_{n-1}$

$k = 1$

$M_0 (M_1 \quad M_{n-1})$

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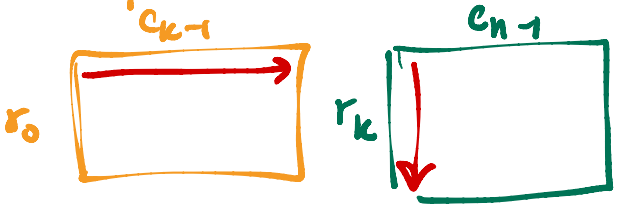

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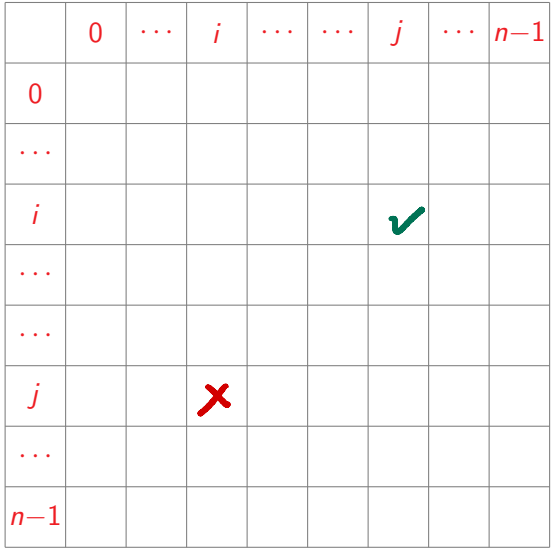
- First factor is $r_0 \times c_{k-1}$, second is $r_k \times c_{n-1}$, where $r_k = c_{k-1}$
- Let $C(0, n-1)$ denote the overall cost
- Final multiplication is $O(r_0 r_k c_{n-1})$
- Inductively, costs of factors are $C(0, k-1)$ and $C(k, n-1)$
- $C(0, n-1) = C(0, k-1) + C(k, n-1) + r_0 r_k c_{n-1}$
- Which k should we choose?
  - Try all and choose the minimum!
- Subproblems?
  - $M_0 \cdot M_1 \cdots M_{k-1}$ would decompose as $(M_0 \cdots M_{j-1}) \cdot (M_j \cdots M_{k-1})$
  - Generic subproblem is $M_j \cdot M_{j+1} \cdots M_k$
- $C(j, k) = \min_{j<\ell\leq k} [C(j, \ell-1) + C(\ell, k) + r_j r_\ell c_k]$
- Base case: $C(j, j) = 0$ for $0 \leq j < n$

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$



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- Compute $C(i,j)$, $0 \leq i,j < n$
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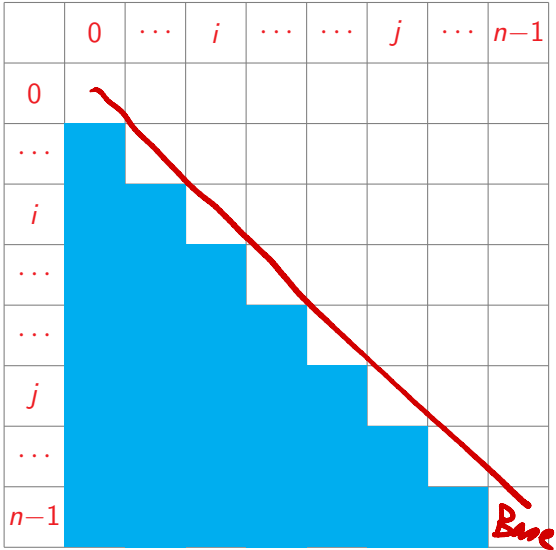

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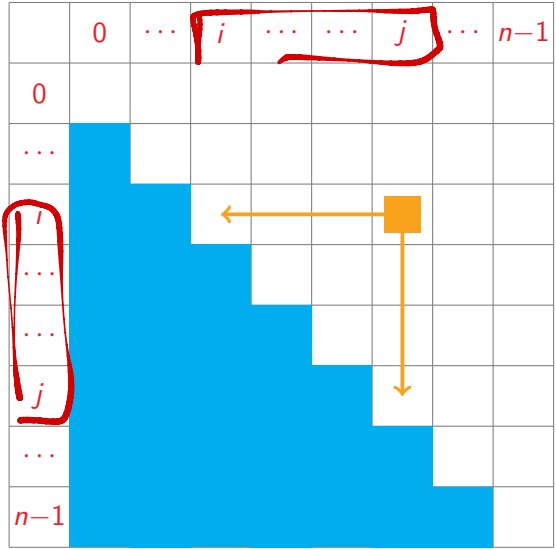

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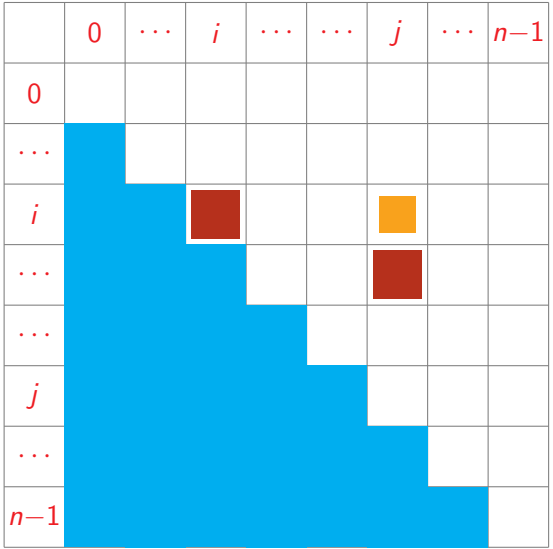

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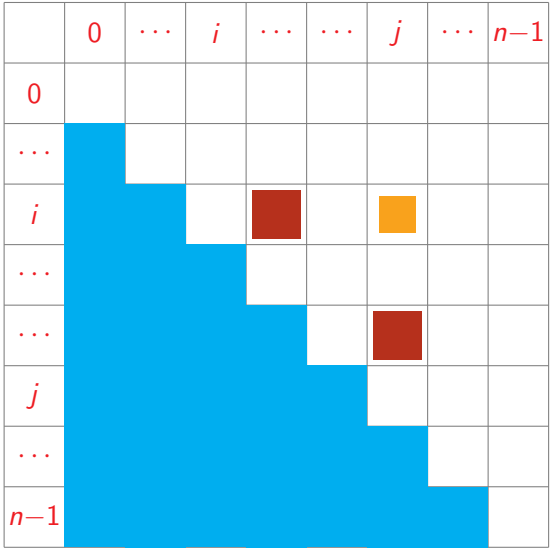

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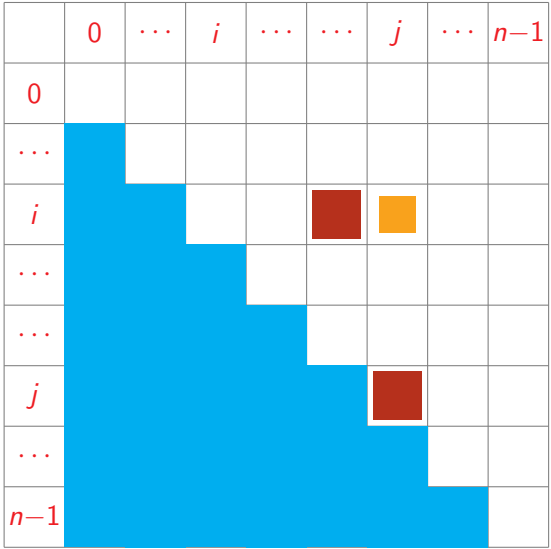

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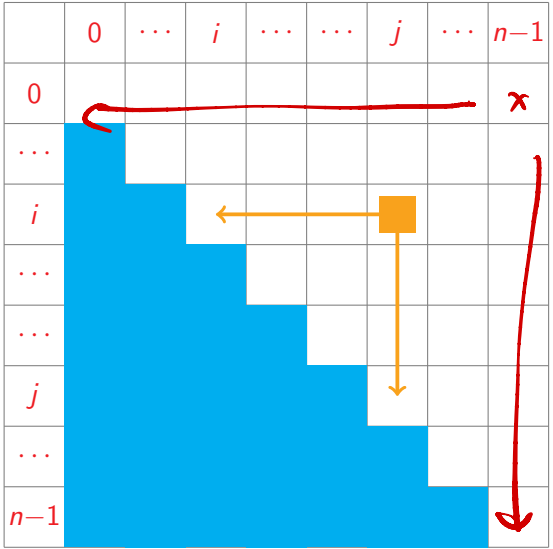

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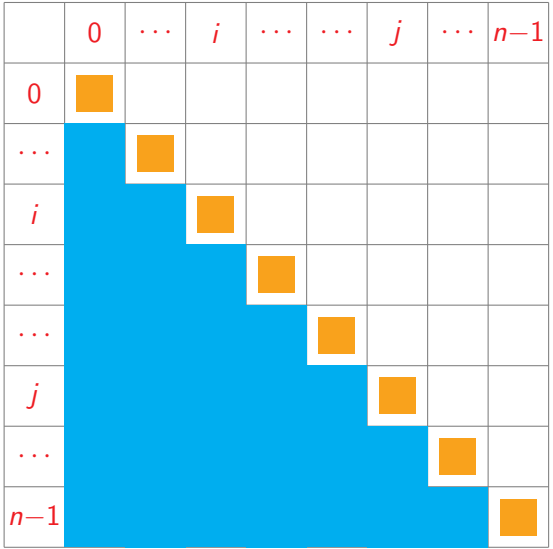

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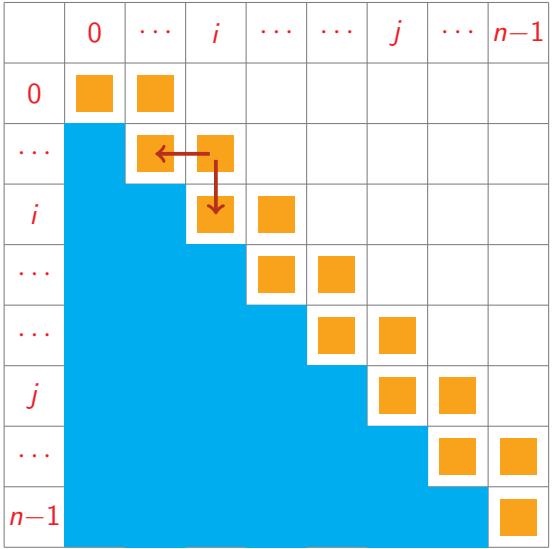

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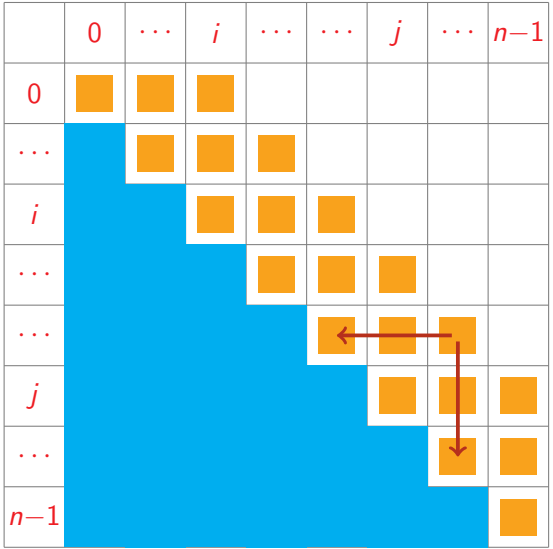

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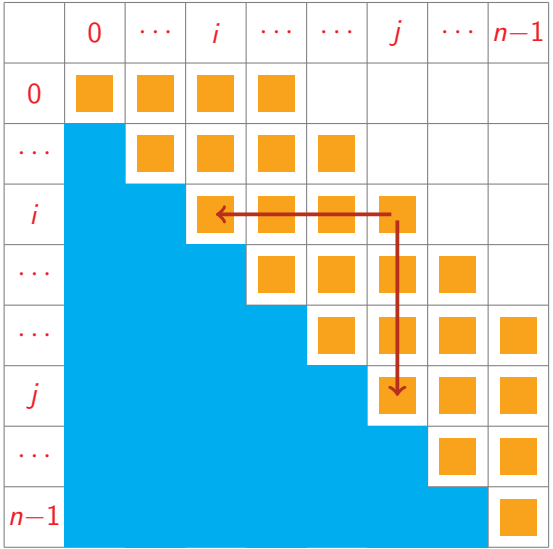

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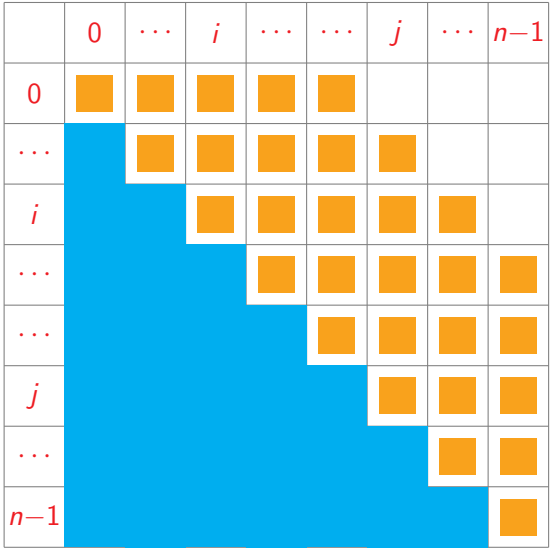

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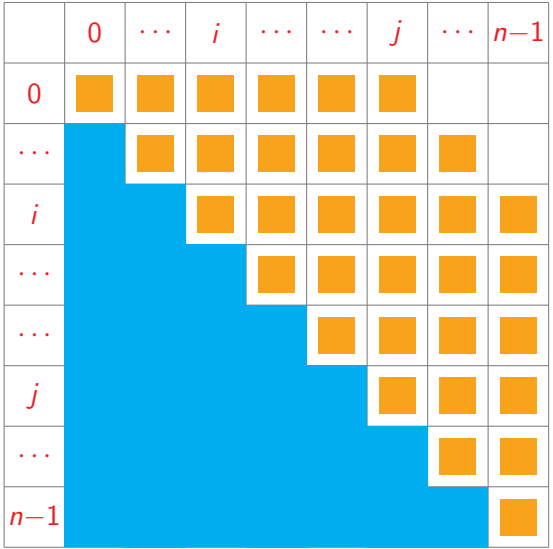

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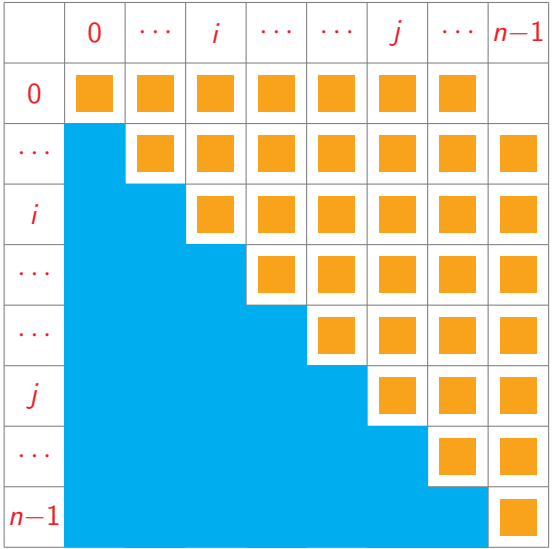

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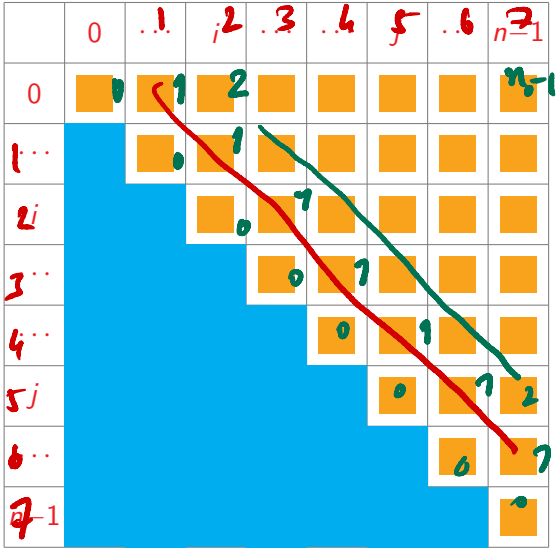

Implementation

```
def C(dim):
  # dim: dimension matrix,
  #      entries are pairs (r_i,c_i)
  import numpy as np
  n = dim.shape[0]
  C = np.zeros((n,n))
  for i in range(n):
    C[i,i] = 0
  for diff in range(1,n):
    for i in range(0,n-diff):
      j = i + diff
      C[i,j] = C[i,i] +
               C[i+1,j] +
               dim[i][0]*dim[i+1][0]*dim[j][1]
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  return(C[0,n-1])
```
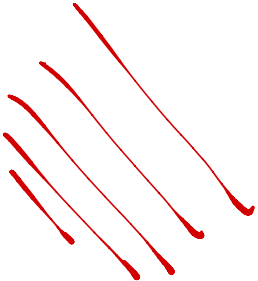

Base Case



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Complexity

Implementation

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Complexity

- We have to fill a table of size $O(n^2)$

Implementation

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Complexity

- We have to fill a table of size $O(n^2)$
- Filling each entry takes $O(n)$

Implementation

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Complexity

- We have to fill a table of size $O(n^2)$
- Filling each entry takes $O(n)$
- Overall, $O(n^3)$

Longest ascending subsequence

How?

Can we use LCS in any way?

LCS (Sort(L), L) ⇒ ?

n^2

$n \log n$ to sort

n^2 for LCS

Longest ascending subsequence

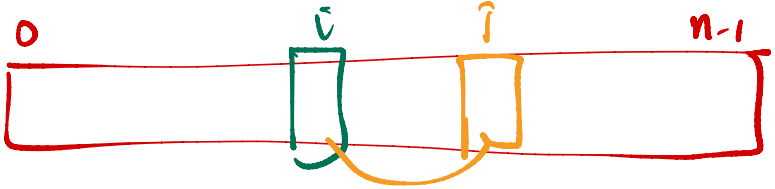
Direct inductive solution

$a_0\ a_1\ a_2\ \cdots\ a_{n-1}$

LLAS[i] — length of LAS in $a_i\ a_{i+1}\ \cdots\ a_{n-1}$ or $a_0\ a_1\ \cdots\ a_i$ ✓

LLAS[i] is longest ascending subsequence starting at a_i

Longest ascending subsequence



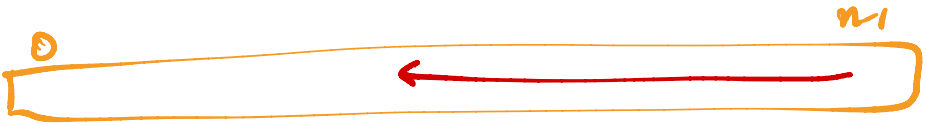
$$LLAS[i] = 1 + \max \{ LLAS[j] \mid j > i, L[j] \geq L[i] \}$$

Cost of $LLAS[i] = O(n-i)$

Longest ascending subsequence

Base Case

LAS[n-1] = 1



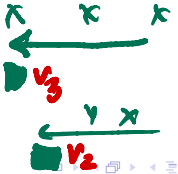
$= O(n^2)$

Longest ascending subsequence

To my right I have maximal ascending subsequences of many lengths

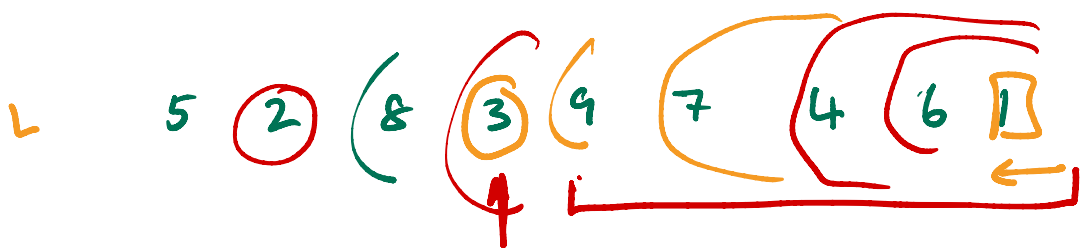
LIAS of length 2

LIAS of length 3



$V_3 \leq V_2$

maxval[k] = smallest value in AS of length k



	1	2	3	4	5	6	7	8	9
max	~~1~~ ~~6~~ ~~9~~	~~4~~ 8	~~3~~ 5	2	0	0	0	.	.

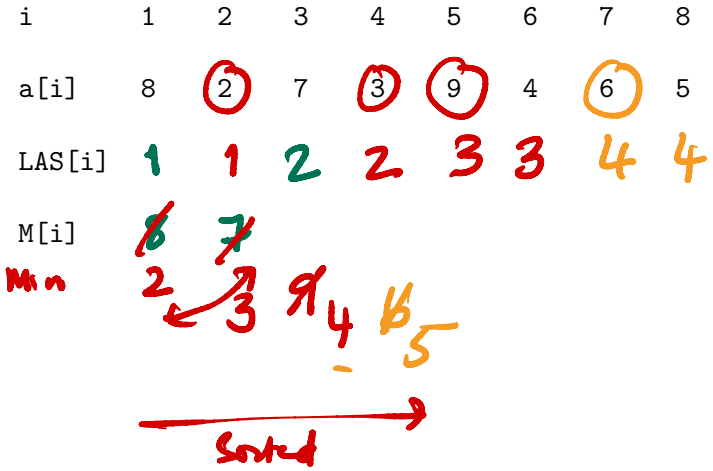
To compute LLAS[i]

Binary search on maxval to find largest k such that $L[i] \leq maxval[k]$

$$LLAS[i] = 1 + k$$

$$maxval[k+1] = \max(maxval(k+1), L[i])$$

Longest ascending subsequence



Cost

Each entry requires a binary search on maxval (or minval)

$n \cdot \log n$