Interprocedural analysis: Sharir-Pnueli’s functional approach

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We want join over all “valid” paths at each program point.

Simply taking “JOP” on extended CFG would lose precision.

Can we compute “JVP” (Join over Valid Paths) values instead?

- JOP
- JVP (interprocedurally valid)
Example program: Available expressions analysis

- 0 (not available)
- 1 (available)
- ⊥

Lattice for Av-Exp analysis.

Is \(a \cdot b\) available at program point \(N\)?

read \(a, b\)

\[t := a \cdot b\]

call \(p\)

\[t := a \cdot b\]

print \(t\)
Example program: Available expressions analysis

- 0 (not available)
- 1 (available)
- \perp

Lattice for Av-Exp analysis.

- Is \(a*b\) available at program point \(N\)?
- No if we consider all paths.
Example program: Available expressions analysis

- 0 (not available)
- 1 (available)
- ⊥

Lattice for Av-Exp analysis.

- Is \(ab\) available at program point \(N\)?
- No if we consider all paths.
- Yes if we consider interprocedurally valid paths only.
Convention: $r_p$ and $e_p$ are respectively the root and return nodes of procedure $p$. Root of the main procedure is $r_1$.

A path $\rho$ is interprocedurally valid and complete if the sequence of call nodes and return notes form a balanced parenthesis string.

A path in $IVP_0(r_1, D)$ for example program:

- $C$ ("call $p$") $\cdot$ $O$ $\cdots$ $H$ ("call $p$") $\cdot$ $L$ $\cdots$ $J$ ("ret") $\cdot$ $M$ $\cdots$ $J$ ("ret") $\cdot$ $N$ $\cdot$ $D$.

Note that "call $p$" must be matched by "ret$_p$."
A path $\rho$ is **interprocedurally valid** if it is a prefix of a valid and complete path.

A path in $\text{IVP}(r_1, I)$ for example program:
Defining JVP

For a given program $P$ and analysis $((D, \leq), f_{MN}, d_0)$, the join over all interprocedurally valid paths (JVP) at point $N$ is defined to be:

$$\bigcup_{\rho \in IVP(r_1, N)} f_\rho(d_0).$$
In non-procedural case, we setup equations to capture JOP assuming distributivity. Least solution to these equations gave us exact/over-approx JOP depending on distributive/monotonic framework.

Try to set up similar equations for $x_N$ (JVP at program point $N$).
Instead try to capture join over complete paths first

- Set up equations to capture join over complete paths.
- Now set up equations to capture JVP using join over complete path values.
**Basic idea: Why join over complete paths help**

An IVP path $\rho$ from $r_1$ to $N$ in procedure $p$ can be written as $\delta \cdot \eta$ where $\delta$ is in $\text{IVP}(r_1, r_p)$, and $\eta$ is in $\text{IVP}_0(r_p, N)$.

Consider point where procedure $p$ was last entered.
For a procedure \( p \) and node \( N \) in \( p \), define:

\[
\phi_{r_p, N} : D \rightarrow D
\]

given by

\[
\phi_{r_p, n}(d) = \bigcup \text{paths } \rho \in \text{IVP}_0(r_p, N) f_{\rho}(d).
\]

\( \phi_{r_p, N} \) is thus the join of all functions \( f_{\rho} \) where \( \rho \) is an interprocedurally valid and complete path from \( r_p \) to \( N \).
Equations (1) to capture $\phi_{r_p,N}$

\[
\begin{align*}
\psi_{r_p,r_p} &= id_D \\
\psi_{r_p,N} &= f_{MN} \circ \psi_{r_p,M} \\
\psi_{r_p,N} &= \psi_{r_p,e_q} \circ \psi_{r_p,M} \\
\psi_{r_p,N} &= \psi_{r_p,L} \sqcup \psi_{r_p,M}.
\end{align*}
\]
Example: Equations for $\phi$’s

\[
\begin{align*}
\psi_{A,A} &= \text{id} \\
\psi_{A,B} &= 0 \circ \psi_{A,A} \\
\psi_{A,C} &= 1 \circ \psi_{A,B} \\
\psi_{A,P} &= \phi_{F,J} \circ \psi_{A,C} \\
\psi_{A,D} &= 1 \circ \psi_{A,P} \\
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\phi_{F,F} &= \text{id} \\
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\phi_{F,I} &= 1 \circ \psi_{F,Q} \\
\phi_{F,J} &= \psi_{F,I} \cup \psi_{F,K}
\end{align*}
\]
Equations (2) to capture JVP

\[
\begin{align*}
x_1 & \geq d_0 \\
x_{r_p} & = \bigcup_{\text{calls } c \text{ to } p \text{ in } q} \phi_{r_q,c}(x_{r_q}) \\
x_n & = \phi_{r_p,n}(x_{r_p}) \quad \text{for } n \in \mathbb{N}_p - \{r_p\}.
\end{align*}
\]
Example: Equations for $x_N$’s (JVP)

\[
\begin{align*}
  x_A & \geq 0 \\
  x_B & = 0(x_A) \\
  x_C & = 1(x_A) \\
  x_P & = 1(x_A) \\
  x_D & = 1(x_A) \\
  x_E & = 1(x_A) \\
  x_F & = 1(x_A) \sqcup 0(x_F) \\
  x_G & = id(x_F) \\
  x_K & = id(x_F) \\
  x_H & = 0(x_F) \\
  x_Q & = 0(x_F) \\
  x_I & = 1(x_F) \\
  x_J & = id(x_F).
\end{align*}
\]

Fig. shows values of $\phi_{r_p, N}$’s in bold.
Correctness and algo

- Consider lattice \((F, \leq)\) of functions from \(D\) to \(D\), obtained by closing the transfer functions, identity, and \(f_\perp : d \mapsto \perp\) (denoted \(f_\Omega\) by Sharir-Pnueil) under composition and join.
- Ordering is \(f \leq g\) iff \(f(d) \leq g(d)\) for each \(d \in D\).
- \((F, \leq)\) is also a complete lattice.
- \(\overline{f}\) induced by Eq (1) is a monotone function on the complete lattice \((\overline{F}, \overline{\leq})\).
- LFP / least solution exists.

Claim

\(\phi_{r_p, N}'s\) are the least solution to Eq (1) when \(f_{MN}'s\) are distributive. Otherwise \(\phi_{r_p, N}'s\) are dominated by the least solution to Eq (1).

Kleene/Kildall’s algo will compute LFP (assuming \(D\) finite).
Correctness and algo - II

- \( \overline{f} \) induced by Eq (2) is a monotone function on the complete lattice \((\overline{D}, \leq)\).
- LFP / least solution exists.

**Claim**

JVP\(_N\)'s are the least solution to Eq (2) when \( f_{MN} \)'s are distributive. Otherwise JVP\(_N\)'s are dominated by the least solution to Eq (2).

Kleene/Kildall’s algo will compute LFP (assuming \( D \) finite).
Example: Equations for $\phi$’s

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\begin{align*}
\psi_{A,A} &= id \\
\psi_{A,B} &= 0 \circ \psi_{A,A} \\
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\psi_{A,P} &= \phi_F, J \circ \psi_{A,C} \\
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\end{align*}
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Motivation Functional Approach

Example

Exercise 1 Iterative Approach

Example: Equations for φ’s

\[ \psi_{A,A} = \text{id} \]
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Diagram:

- Node A: \( \text{id} \)
- Node B: 0
- Node C: 1
- Node D: \( f_\perp \)
- Node E: \( f_\perp \)
- Node F: \text{id}
- Node G: \text{id}
- Node H: 0
- Node I: 1
- Node J: \( f_\perp \)
- Node K: \text{id}
- Node L: \text{id}
- Node M: \text{id}
- Node N: \text{id}
- Node O: 6
- Node P: \text{id}
- Node Q: \text{id}
- Node R: \text{id}
- Node S: \text{id}
- Node T: \text{id}
- Node U: \text{id}
- Node V: \text{id}
- Node W: \text{id}
- Node X: \text{id}
- Node Y: \text{id}
- Node Z: \text{id}

Arrows:
- \( r_1 \)
- \( r_2 \)
- \( c_1 \)
- \( c_2 \)
- \( n_1 \)
- \( n_2 \)
- \( e_1 \)
- \( e_2 \)

Actions:
- \( \text{read a,b} \)
- \( \text{call p} \)
- \( \text{print t} \)
- \( \text{ret} \)
Example: Equations for $\phi$'s

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Example: Equations for $\phi$’s

$\psi_{A,A} = id$

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Motivation

Functional Approach

Example

Exercise 1

Iterative Approach
Example: Equations for $\phi$’s

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Example: Equations for $x_N$’s (JVP)

\[
\begin{align*}
xA & \geq 0 \\
xB &= 0(x_A) \\
xC &= 1(x_A) \\
XP &= 1(x_A) \\
XD &= 1(x_A) \\
XE &= 1(x_A) \\
XF &= 1(x_A) \sqcup 0(x_F) \\
XG &= id(x_F) \\
XK &= id(x_F) \\
XH &= 0(x_F) \\
XQ &= 0(x_F) \\
XI &= 1(x_F) \\
XJ &= id(x_F).
\end{align*}
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**Example: Equations for $x_N$'s (JVP)**

- $x_A \geq 0$
- $x_B = 0(x_A)$
- $x_C = 1(x_A)$
- $x_P = 1(x_A)$
- $x_D = 1(x_A)$
- $x_E = 1(x_A)$
- $x_F = 1(x_A) \sqcup 0(x_F)$
- $x_G = id(x_F)$
- $x_K = id(x_F)$
- $x_H = 0(x_F)$
- $x_Q = 0(x_F)$
- $x_I = 1(x_F)$
- $x_J = id(x_F)$

Fig shows initial (red) and final (blue) values.
Example: Equations for $x_N$’s (JVP)

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    x_A & \geq 0 \\
    x_B & = 0(x_A) \\
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    x_D & = 1(x_A) \\
    x_E & = 1(x_A) \\
    x_F & = 1(x_A) \uplus 0(x_F) \\
    x_G & = id(x_F) \\
    x_K & = id(x_F) \\
    x_H & = 0(x_F) \\
    x_Q & = 0(x_F) \\
    x_I & = 1(x_F) \\
    x_J & = id(x_F). \\
\end{align*}

Fig shows initial (red) and final (blue) values.
Exercise: Use the functional method to do interprocedural constant propagation analysis for the program below, with initial value $\emptyset$. 

```
1. $a := a + 1$
2. $a := 0$
3. call p
4. print a
5. G
6. cond
7. G
8. $a := a + 1$
9. call p
10. $a := a - 1$
11. ret
```

Use the program structure to propagate constants through the program.
Summary of functional approach

- Uses a two step approach
  1. Compute $\phi_{r_p,N}$’s.
  2. Compute $x_n$’s (JVP’s) at each point.

Summary of conditions: For each property (column heading), the conjunction of the ticked conditions (row headings) are sufficient to ensure the property.

<table>
<thead>
<tr>
<th></th>
<th>Termination</th>
<th>Least Sol of Eq(2) $\geq$ JVP</th>
<th>Least Sol of Eq(2) = JVP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{MN}$’s monotonic</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Finite underlying lattice</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Distributive</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>
Iterative/Tabulation Approach

- Maintain a table of values representing the current value of \( \phi_{r_p,N} \) for each program point \( N \) in procedure \( p \).
- Informally, at \( N \) in procedure \( p \), the table has an entry \( d \mapsto d' \) if we have seen valid paths \( \rho \) from \( r_1 \) to \( r_p \) with \( \bigcup \rho f_{\rho}(d_0) = d \), and valid and complete paths \( \delta \) from \( r_p \) to \( N \) with \( \bigcup \delta f_{\delta}(d) = d' \).
- Apply Kildall’s algo with initial value of \( d_0 \mapsto d_0 \) at \( r_1 \).
Propogation rules

- If \( d \mapsto d' \) at point \( M \), and statement corresponding to \( MN \) is not a call or \texttt{ret}, then propagate \( d \mapsto f_{MN}(d') \) to point \( N \).
- If \( d \mapsto d' \) at point \( M \), and statement after \( M \) is call \( q \), then
  - propagate \( d \mapsto d' \) to point \( r_q \),
  - propagate \( d \mapsto d'' \) to return site of \( N \) of \( M \), provided we have \( d' \mapsto d'' \) at point \( e_q \).
- If \( d \mapsto d' \) at point \( e_q \) (i.e. before \texttt{ret} in procedure \( q \)), then
  - If \( LN \) corresponds to a call \( q \) and \( d'' \mapsto d \) at \( L \), then
    - propagate \( d'' \mapsto d' \) to point \( N \). (Do this for all such \( N \).)
Example: Computing $\phi$’s iteratively: 1

Motivation

Iterative Approach

Example

Exercise 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1

Iterative Approach

Example: Computing $\phi$’s iteratively: 1
Example: Computing \( \phi \)'s iteratively: 2

Motivation

Functional Approach

Example

Exercise 1

Iterative Approach

```
read a, b

\[ t := a \times b \]

call p

\[ t := a \times b \]

print t

```

```
\begin{align*}
a &:= a - 1 \\
\text{call p} \\
\text{print } t \\
\text{ret}
\end{align*}
```
Example: Computing $\phi$'s iteratively: 3

Iterative Approach

Example Exercise 1

Motivation

Functional Approach

Example

Exercise 1

Iterative Approach
Example: Computing $\phi$'s iteratively: 4

Iterative Approach Example Exercise 1

Iterative Approach

Motivation Functional Approach Example Exercise 1

Iterative Approach

Example: Computing $\phi$'s iteratively: 4

Iterative Approach Example Exercise 1

Iterative Approach

Example: Computing $\phi$'s iteratively: 4

Iterative Approach Example Exercise 1

Iterative Approach
Example: Computing $\phi$’s iteratively: 5

Iterative Approach

Example: Computing $\phi$’s iteratively: 5

Motivation Functional Approach Example Exercise 1 Iterative Approach

read a, b

\[ \text{call p} \]

\[ \text{print t} \]

\[ t := a \times b \]

\[ t := a \times b \]

\[ \text{call p} \]

\[ a == 0 \]

\[ \text{call p} \]

\[ a := a-1 \]

\[ t := a \times b \]

\[ \text{print t} \]

\[ \text{ret} \]
Example: Computing $\phi$’s iteratively: 6

Motivation

Iterative Approach

Example

Exercise 1
Example: Computing $\phi$’s iteratively: 7

```
read a, b

\[ t := a \times b \]

\[ a := a - 1 \]

\[ \text{call } p \]

\[ t := a \times b \]

\[ \text{print } t \]
```

```
\[ t := a \times b \]

\[ a := a - 1 \]

\[ \text{call } p \]

\[ t := a \times b \]

\[ \text{ret} \]
```
Example: Computing $\phi$’s iteratively: 8

1. **Iterative Approach**

Example: Computing $\phi$’s iteratively:

- $a := a - 1$
- $t := a * b$
- $a := a - 1$
- $t := a * b$
- ...
Example: Computing $\phi$'s iteratively: 9

```
read a, b

A 0

B 0

t := a*b

C 1

call p

P 1

t := a*b

D 1

print t

E 1
```

```
a := a-1

F 0 1

G 0 1

a == 0

H 0 0

call p

I

t := a*b

J

ret
```

```
K 0 1
```

```
L

M

N

O
```

```
F 6
```

```
G 0 1
```

```
a := a-1
```

```
H 0 0
```

```
I
```

```
J
```

```
K 0 1
```

```
L

M

N

O
```

```
F 6
```

```
G 0 1
```

```
a := a-1
```

```
H 0 0
```

```
I
```

```
J
```

```
K 0 1
```
Example: Computing $\phi$'s iteratively: 10

```
a := a-1
F
G

read a, b
B

A

0

C

1

call p

P

1

t := a*b

D

1

call p

I

0

J

0

K

0

L

0

M

0

N

0

O

0

F

6

G

a == 0

H

0

0

K

0

1

L

O

1

M

1

N

1

ret

E

1

t := a*b

D

1

print t

E

1
```
Example: Computing $\phi$’s iteratively: 11

```
read a, b

A 0

B 0

t := a*b

C 1

call p

P 1

t := a*b

D 1

print t

E 1

I 0

J 0

ret
```

```
a == 0

G 0

H 0

call p

Q 0

K 0
```

```
a := a-1

O

F 0

6
```

```
t := a*b

M

N
```

```
0 1
92x178
```

```
1
92x165
0
```

```
-0
92x131
-1
92x62
0 0
```

```
0 1
92x104
```

```
0 1
92x573
```
Example: Computing $\phi$’s iteratively: 12
Example: Computing $\phi$’s iteratively: 13

```
read a, b

t := a * b

call p

print t
```

```
a := a - 1

call p

t := a * b

ret
```
Example: Finally compute $x_N$’s from $\phi$ values

At each point $N$ take join of reachable $\phi_{r_p,N}$ values.

```
read a, b

A

B

t := a * b

C

call p

P

t := a * b

D

print t

E
```

```
a := a - 1

F

G

a == 0

H

a := a - 1

I

call p

J

t := a * b

K

L

N

M

O

ret
```

Correctness of iterative algo

\[
\begin{align*}
    x_1 & \geq d_0 \\
    x_{r_p} &= \bigcup_{\text{calls } c \text{ to } p \text{ in } q} \psi_{r_q,c}^*(x_{r_q}) \\
    x_n &= \psi_{r_p,n}^*(x_{r_p}) \quad \text{for } n \in \mathbb{N}_p - \{r_p\}.
\end{align*}
\]

- Iterative algo terminates provided underlying lattice is finite.
- It computes the least solution to the equations above, where \(\psi^*(r_{p_N})\)'s are the least solution to Eq (1).
- It thus computes an overapproximation of JVP for monotonic transfer functions, and exact JVP when transfer functions are distributive.
Exercise 2: Iterative algo

Exercise: Use the iterative algo to do constant propagation analysis for the program below with initial value $∅$:

```
A := 0
B
Call p
C
Print a
D
```

```
cond
G
H
Call p
P
I
J
Ret
```

```
a := a + 1
```

```
a := a - 1
```
Comparing functional vs iterative approach

- Functional algo can terminate even when underlying lattice is infinite, provided we can represent and compose/join functions “symbolically”.
- Iterative is typically more efficient than functional since it only computes $\phi_{r_p,N}$’s for values reachable at start of procedure.