Typed λ-calculus

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Programming Language Concepts

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Adding types to λ-calculus

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- The first functional programming language, **LISP**, was also untyped
- Modern languages such as **Haskell**, **ML**, ...are typed
- What is the theoretical foundation for such languages?

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  - We will only learn Curry typing

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  - `data BTree a = Nil | Node (BTree a) a (BTree a)`

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- When constructing expressions, build up the type from the types of the parts

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  - $p \to (q \to p)$
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  - $(p \to p) \to (p \to q)$
- $\sigma, \tau, \ldots$ stand for arbitrary types
- $\to$ is right associative: $\sigma \to \tau \to \theta$ is short for $\sigma \to (\tau \to \theta)$

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- The typing rules:

$$\Gamma, x : \tau \vdash x : \tau \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x \cdot M) : \sigma \rightarrow \tau} \qquad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash (MN) : \tau}$$

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Curry typing: Examples

- $$\frac{x : p \vdash x : p}{\vdash \lambda x \cdot x : p \to p}$$

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- $$\frac{x : p \vdash x : p}{\vdash \lambda x \cdot x : p \rightarrow p}$$

- $$\frac{\dfrac{x : p, y : q \vdash x : p}{x : p \vdash \lambda y \cdot x : q \rightarrow p}}{\vdash \lambda xy \cdot x : p \rightarrow (q \rightarrow p)}$$

Curry typing: Examples

- Let $\Gamma = \{x : p \to q \to r, y : p \to q, z : p\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash x : p \to q \to r \quad \Gamma \vdash z : p}{\Gamma \vdash xz : q \to r} \quad \dfrac{\Gamma \vdash y : p \to q \quad \Gamma \vdash z : p}{\Gamma \vdash yz : q}}{\Gamma \vdash xz(yz) : r}}{x : p \to q \to r, y : p \to q \vdash \lambda z \cdot xz(yz) : p \to r}}{x : p \to q \to r \vdash \lambda yz \cdot xz(yz) : (p \to q) \to (p \to r)}}{\vdash \lambda xyz \cdot xz(yz) : (p \to q \to r) \to (p \to q) \to (p \to r)}$$

Curry typing: Examples

- Let $\Gamma = \{f : q, x : p\}$

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- Let $\Delta = \{f : p \to p, x : p\}$

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- Define **int** $:= (p \to p) \to (p \to p)$
- For all $n \in \mathbb{N}$, $\vdash$ «n» : **int**

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- $\textbf{succ}$ can be given the type $\textbf{int} \longrightarrow \textbf{int}$
- Let $\Gamma = \{m : \textbf{int}, f : p \longrightarrow p, x : p\}$

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- But we can derive the judgement $«m»\, «n»$: **int**
- For example, letting $\tau := p \rightarrow p$,

$$\frac{\vdash «2» : (\tau \rightarrow \tau) \rightarrow (\tau \rightarrow \tau) \qquad \vdash «2» : (p \rightarrow p) \rightarrow (p \rightarrow p)}{\vdash «2»\, «2» : \mathbf{int}}$$

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- f is definable in typed λ-calculus iff it is essentially a polynomial function!

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- Extend $\longrightarrow_\beta$ to one-step reduction $\longrightarrow$, as usual
- Extend to many-step $\stackrel{*}{\longrightarrow}_\beta$ as usual
- $\stackrel{*}{\longrightarrow}_\beta$ is Church-Rosser
  - Same proof as for untyped λ-calculus

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  - **(weakly) normalizing** if it has a normal form
    - **Example:** $(\lambda x \cdot y)\Omega$
    - **Counterexample:** Ω
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- **Principal type**
  - a type for a term M such that every other type for M is got by uniformly replacing each variable by a type
  - unique for each typable term – modulo renaming of variables!