On reduction

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Programming Language Concepts

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One step reduction

- Can have other reduction rules like β
- Observe that if x does not occur free in M, then

$$\text{for all } N, (\lambda x \cdot (Mx))N \longrightarrow_\beta MN$$

- Thus $\lambda x \cdot (Mx)$ behaves just like M
- New reduction rule η (when $x \notin \mathbf{fv}(M)$)

$$\lambda x \cdot (Mx) \longrightarrow_\eta M$$

One step reduction

- Define a one step reduction inductively (where $x \in \{\beta, \eta, \ldots\}$)

$$\frac{M \longrightarrow_x M'}{M \longrightarrow M'}$$

$$\frac{M \longrightarrow M'}{MN \longrightarrow M'N} \qquad \frac{N \longrightarrow N'}{MN \longrightarrow MN'} \qquad \frac{M \longrightarrow M'}{\lambda x \cdot M \longrightarrow \lambda x \cdot M'}$$

Multistep reduction and equivalence

- $M \xrightarrow{*} N$: repeatedly apply $\longrightarrow$ to get N
  - There is a sequence $M = M_0, M_1, \ldots, M_k = N$ such that for each $i < k : M_i \longrightarrow M_{i+1}$
- $M \longleftrightarrow N$: M is **equivalent** to N
  - There is a sequence $M = M_0, M_1, \ldots, M_k = N$ such that for each $i < k$: either $M_i \longrightarrow M_{i+1}$ or $M_{i+1} \longrightarrow M_i$

Normal forms

- Computation — a maximal sequence of reduction steps
- **Values** — expressions that cannot be further reduced
- An **expression in normal form** or a **normal term**
- We allow reduction in any context, so multiple redexes may qualify for reduction
  - **Recall**: A **redex** (or **reducible expression**) is a subexpression of the form $(\lambda x \cdot M)N$ (or $\lambda x \cdot (Mx)$, in the case of η-reduction)

Natural questions

- Does every term reduce to a normal form?
- Can a term reduce to more than one normal form, depending on the reduction sequence?
- If a term has a normal form, can we always find it?

Normal forms ...

Does every term reduce to normal form?

- Consider the terms $\omega = \lambda x \cdot xx$ and $\Omega = \omega\omega$
- $\Omega = (\lambda x \cdot xx)(\lambda x \cdot xx) \longrightarrow_\beta (\lambda x \cdot xx)(\lambda x \cdot xx) = \Omega$
  - Reduction never terminates

Normal forms …

Can a term reduce to more than one normal form, depending on the reduction sequence?

- Consider the term **false** $\Omega = (\lambda yz \cdot z)((\lambda x \cdot xx)(\lambda x \cdot xx))$
- **Outermost reduction**

$$(\lambda yz \cdot z)((\lambda x \cdot xx)(\lambda x \cdot xx)) \longrightarrow_\beta \lambda z \cdot z$$

- **Innermost reduction**

$$(\lambda yz \cdot z)((\lambda x \cdot xx)(\lambda x \cdot xx)) \longrightarrow_\beta (\lambda yz \cdot z)((\lambda x \cdot xx)(\lambda x \cdot xx))$$

- Choice of reduction strategy may determine whether a normal form can be reached, but can more than one normal form be reached?

Normal forms …

If a term has a normal form, can we always find it?

- **Yes!** We can do a breadth-first search of the reduction graph, and we are guaranteed to find a normal form eventually
- We could also reduce the term following the strategy of **leftmost outermost reduction**
- If a term has a normal form, leftmost outermost reduction will find it!

Normal forms …

Given a term, can we determine if it has a normal form?

- We have seen how to encode recursive functions in the λ-calculus
- We cannot in general determine if the computation of $f(n)$ terminates, given f and n
- But computing $f(n)$ is equivalent to finding the normal form of $\mathbf{f}$ «n»
- So $f(n)$ is defined iff $\mathbf{f}$ «n» has a normal form
- So checking whether a given term has a normal form is **undecidable**

Church-Rosser theorem

Theorem (Church-Rosser)

If $M \longleftrightarrow N$ *there is a term* P *such that* $M \stackrel{*}{\longrightarrow} P$ *and* $N \stackrel{*}{\longrightarrow} P$

- **Question**: Can a term reduce to more than one normal form, depending on the reduction sequence?
- **Answer**: No!
  - Suppose a term M_0 reduces to two normal forms M and N
  - Then $M \longleftrightarrow N$
  - Thus there is a P such that $M \stackrel{*}{\longrightarrow} P$ and $N \stackrel{*}{\longrightarrow} P$ (by Church-Rosser)
  - But since M and N are already in normal form, $M = P = N$ (upto renaming of bound variables)

Church-Rosser theorem

Theorem (Church-Rosser)

If $M \longleftrightarrow N$ there is a term P such that $M \stackrel{*}{\longrightarrow} P$ and $N \stackrel{*}{\longrightarrow} P$

Proof.

- **Recall**: $M \longleftrightarrow N$ iff there is a sequence $M = M_0, M_1, \ldots, M_k = N$ such that for all $i < k$: either $M_i \longrightarrow N$ or $N \longrightarrow M$
- **Claim**: For all $i \leqslant k$, there is a P_i such that $M_0 \stackrel{*}{\longrightarrow} P_i$ and $M_i \stackrel{*}{\longrightarrow} P_i$
  - **Base case**: Choose $P_0 = M_0$
  - **Induction case**: Suppose there is a P_i such that $M_0 \stackrel{*}{\longrightarrow} P_i$ and $M_i \stackrel{*}{\longrightarrow} P_i$
  - If $M_{i+1} \longrightarrow M_i$, take $P_{i+1} = P_i$
  - If $M_i \longrightarrow M_{i+1}$, use the **Diamond property** to arrive at the desired P_{i+1} ♣

Church-Rosser theorem

Theorem (Diamond property)

If $M_0 \xrightarrow{*} M$ and $M_0 \xrightarrow{*} N$, there is a term P such that $M \xrightarrow{*} P$ and $N \xrightarrow{*} P$

- We can talk of the Diamond property for any relation R
- R has the Diamond property if

$$(\forall a, b, c)[(aRb \wedge aRc) \Rightarrow (\exists d)(bRd \wedge cRd)]$$

Proposition

If R has the Diamond property, so does R^*

The proof is by induction on length of R-chains

Diamond property

Theorem (Diamond property)

If $M_0 \xrightarrow{*} M$ and $M_0 \xrightarrow{*} N$, there is a term P such that $M \xrightarrow{*} P$ and $N \xrightarrow{*} P$

Proposition

If R has the Diamond property, so does R^*

Unfortunately, $\longrightarrow$ does not have the Diamond property!

- Recall that $\omega = \lambda x.xx$ and $\mathbf{I} = \lambda x.x$
- $\omega(\mathbf{II}) \longrightarrow (\mathbf{II})(\mathbf{II})$ by outermost reduction and $\omega(\mathbf{II}) \longrightarrow \omega\mathbf{I}$ by innermost reduction
- $\omega\mathbf{I} \longrightarrow \mathbf{II}$ but it takes **two** steps to go from $(\mathbf{II})(\mathbf{II})$ to $\mathbf{II}$!

Diamond property

Solution: Define a new "parallel reduction" $\Longrightarrow$ as follows

$$M \Longrightarrow M \qquad \frac{M \Longrightarrow M'}{\lambda x \cdot M \Longrightarrow \lambda x \cdot M'}$$

$$\frac{M \Longrightarrow M' \quad N \Longrightarrow N'}{MN \Longrightarrow M'N'} \qquad \frac{M \Longrightarrow M' \quad N \Longrightarrow N'}{(\lambda x \cdot M)N \Longrightarrow M'[x := N']}$$

- It is easily shown that
  - if $M \longrightarrow_\beta N$ then $M \Longrightarrow N$
  - if $M \Longrightarrow N$ then $M \stackrel{*}{\longrightarrow}_\beta N$
  - Hence $M \stackrel{*}{\Longrightarrow} N$ iff $M \stackrel{*}{\longrightarrow}_\beta N$
- It can also be shown that $\Longrightarrow$ has the Diamond property

Diamond property

- $M \overset{*}{\Longrightarrow} N$ iff $M \overset{*}{\longrightarrow}_\beta N$
- It can also be shown that $\Longrightarrow$ has the Diamond property
- Hence $\overset{*}{\Longrightarrow}$ (and therefore $\overset{*}{\longrightarrow}_\beta$) has the Diamond property
  - Can be extended in the presence of $\longrightarrow_\eta$ as well

Proposition

If $M_0 \Longrightarrow M$ and $M_0 \Longrightarrow N$ then there is a P such that $M \Longrightarrow P$ and $N \Longrightarrow P$

Proof.

- For every M, define M^*, the term obtained by one application of “maximal” parallel reduction
- Whenever $M \Longrightarrow N$, $N \Longrightarrow M^*$ ♣