Lambda calculus

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Programming Language Concepts

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λ-calculus

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  - How are outputs computed from inputs?

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where $x \in Var$ and $M, N \in \Lambda$.

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  - Apply the function M to the argument N

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- Functions manipulate meaningful data to yield meaningful data
- Can also apply functions to non-meaningful data, but the result has no significance

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  - $\lambda f \cdot (\lambda u \cdot f(uu))(\lambda u \cdot f(uu))$ is short for $(\lambda f \cdot ((\lambda u \cdot f(uu))(\lambda u \cdot f(uu))))$

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- β is the **only** rule we need
- MN is meaningful only if M is of the form $\lambda x \cdot P$
  - Cannot do anything with terms like xx or $(y(\lambda x \cdot x))(\lambda y \cdot y)$

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  - **Warning**: Possible for a variable to be both in $\mathbf{fv}(M)$ and $\mathbf{bv}(M)$

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- The y substituted for x has been "confused" with the y bound by λy

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  - Makes the definition deterministic

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    - There is a sequence $M_0, M_1, \ldots, M_k$ such that

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- In λ-calculus, we encode n by the number of times we apply a function (**successor**) to an element (**zero**)

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- $\mathbf{n}\, gy = (\lambda fx \cdot f(\cdots(fx)\cdots))\, gy \xrightarrow{*}_{\beta} g(\cdots(gy)\cdots) = g^n y$