PLC : Lambda Calculus Assignment

Set: April 25, 2024
Due: May 3, 2024, 23.55

General instructions:

1. Submit your solutions as a PDF. The file should be named `<un>.pdf`, where `un` is your username. You can either write on paper and scan as PDF, or write in an iPad / notebook and export to PDF.
2. Properly parenthesize your lambda expressions and use spacing to keep it readable.
3. Recall that the Church encoding of n, denoted $\langle n \rangle$, is the expression $\lambda fx \cdot f^n x$, where for any λ-terms P and Q, $P^0\,Q$ is defined to be just Q, and $P^{i+1}\,Q := P\,(P^i\,Q\,)$.

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1 Untyped lambda calculus

1. Let $\langle exp \rangle := \lambda p\,q\,f\,x \cdot p\,q\,f\,x$. Assuming that $0^0 = 1$, prove that for all $m, n \geqslant 0$,

$$\langle exp \rangle\ \langle m \rangle\ \langle n \rangle \xrightarrow{*}_{\beta} \langle n^m \rangle.$$

   Hint: Prove the following claims in order:

   (a) For all $i \geqslant 0$, for all λ-terms P and Q, $P^{i+1}\,Q = P^i\,(P\,Q\,)$.

   (b) For all $k, l \geqslant 0$ and for all λ-terms P and Q, $(\langle k \rangle\, P)^l\, Q \xrightarrow{*}_{\beta} P^{kl}\,Q$.

   (c) For all $n, m \geqslant 0$ and for all λ-terms P and Q, $\langle n \rangle^m\, P\,Q \xrightarrow{*}_{\beta} P^{n^m}\,Q$.

   (d) Conclude therefore that $\langle exp \rangle\ \langle m \rangle\ \langle n \rangle \xrightarrow{*}_{\beta} \langle n^m \rangle$.

2. Recall that a **redex** is any λ-expression of the form $(\lambda x \cdot M)\,N$. A **normal term** is one which does not contain a redex (as a subexpression). Prove that every normal term M is of the form $\lambda x_1\,x_2\,\ldots\,x_k \cdot y\,M_1\,M_2\,\ldots\,M_l$, where $k, l \geqslant 0$, y is a variable, and $M_1, \ldots, M_l$ are normal terms. (Note that when $k = 0$, the expression is just $y\,M_1\,M_2\,\ldots\,M_l$, and when $l = 0$, the expression is just $\lambda x_1\,x_2\,\ldots\,x_k \cdot y$. Of course, when $k = l = 0$, the expression is just y.)

3. Find a λ-term that encodes the **predecessor function** *pred* defined as follows:

$$pred(0) := 0$$
$$pred(n+1) := n$$

4. Find a λ-term that encodes the **maximum function** *max* defined as follows:

$$max(m,n) := \begin{cases} m & \text{if } m \geqslant n \\ n & \text{otherwise} \end{cases}$$

5. Find a λ-term M such that for all λ-terms F, G and H,

$$MFGH \xrightarrow{*} FH(M(GH)).$$

   Hint: Recall that for any term K, if we define $M := (\lambda x \cdot K(xx))\,(\lambda x \cdot K(xx))$, then $M \longrightarrow KM$. Find an appropriate K and use this fact.

2 Typed lambda calculus

1. Give the most general types for the following λ-terms. Show the steps of applying the type inference algorithm presented in class to arrive at the type.

   (a) $\lambda fx \cdot fx$

   (b) $\lambda fx \cdot f(fx)$

   (c) $\lambda xyz \cdot xyz$

   (d) $\lambda xyz \cdot x(yz)$

   (e) $\lambda xy \cdot x(\lambda z \cdot zy)$

2. Recall from class that we define $\langle int \rangle := (p \to p) \to (p \to p)$ for a fixed type variable p. We also showed in class that for all $n \geqslant 0$, we can derive the typing judgement $\vdash \langle n \rangle : \langle int \rangle$. Recall the encodings for the successor function, addition, and multiplication:

$$\begin{aligned} \langle succ \rangle &:= \lambda mfx \cdot f(mfx) \\ \langle plus \rangle &:= \lambda mnfx \cdot mf(nfx) \\ \langle mult \rangle &:= \lambda mnfx \cdot m(nf)x \end{aligned}$$

   Derive the following typing judgements:

   (a) $\vdash \langle succ \rangle : \langle int \rangle \to \langle int \rangle$

   (b) $\vdash \langle plus \rangle : \langle int \rangle \to \langle int \rangle \to \langle int \rangle$

   (c) $\vdash \langle mult \rangle : \langle int \rangle \to \langle int \rangle \to \langle int \rangle$