λ Calculus: Lecture 5

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PLC, 20 March 2017

Adding types to λ-calculus

- The basic λ-calculus is untyped
- The first functional programming language, LISP, was also untyped
- Modern languages such as Haskell, ML, $\ldots$ are strongly typed
- What is the theoretical foundation for such languages?

Types in functional programming

The structure of types in Haskell

- Basic types—`Int`, `Bool`, `Float`, `Char`
- Structured types

  [Lists] If `a` is a type, so is `[a]`

  [Tuples] If `a1`, `a2`, ..., `ak` are types, so is `(a1,a2,...,ak)`

- Function types
  - If `a`, `b` are types, so is `a -> b`
  - Function with input `a`, output `b`
- User defined types
  - `Data day = Sun | Mon | Tue | Wed | Thu | Fri | Sat`
  - `Data BTree a = Nil | Node (BTree a) a (Btree a)`

Adding types to λ-calculus ...

- Set Λ of untyped lambda expressions is given by

$$\Lambda = x \mid \lambda x.M \mid MM'$$

  where $x \in Var$, $M, M' \in \Lambda$.

- Add a syntax for basic types

- When constructing expressions, build up the type from the types of the parts

Adding types to λ-calculus ...

- Restrict our language to have just one basic type, written as τ
- No structured types (lists, tuples, ...)
- Function types arise naturally ($\tau \rightarrow \tau$, $(\tau \rightarrow \tau) \rightarrow \tau \rightarrow \tau$, ...

"Simply typed" λ-calculus

A separate set of variables Var_s for each type s

Define Λ_s, expressions of type s, by mutual recursion

- For each type s, every variable $x \in Var_s$ is in Λ_s
- If $M \in \Lambda_t$ and $x \in Var_s$ then $(\lambda x.M) \in \Lambda_{s \to t}$.
- If $M \in \Lambda_{s \to t}$ and $N \in \Lambda_s$ then $(MN) \in \Lambda_t$.
  - Note that application must be well typed

β rule as usual

- $(\lambda x.M)N \to_\beta M\{x \leftarrow N\}$
- We must have $\lambda x.M \in \Lambda_{s \to t}$ and $N \in \Lambda_s$ for some types s, t
- Moreover, if $\lambda x.M \in \Lambda_{s \to t}$, then $x \in Var_s$, so x and N are compatible

“Simply typed” λ-calculus . . .

- Extend $\rightarrow_\beta$ to one-step reduction $\rightarrow$, as usual
- The reduction relation $\rightarrow^*$ is Church-Rosser
- In fact, $\rightarrow^*$ satisifies a much strong property

Strong normalization

A λ-expression is

- normalizing if it has a normal form.
- strongly normalizing if every reduction sequence leads to a normal form

Examples

- $(\lambda x.xx)(\lambda x.xx)$ is not normalizing
- $(\lambda yz.z)((\lambda x.xx)(\lambda x.xx))$ is not strongly normalizing.

Strong normalization ...

A λ-calculus is strongly normalizing if every term in the calculus is strongly normalizing

Theorem

The simply typed λ-calculus is strongly normalizing

Proof intuition

- Each β-reduction reduces the type complexity of the term
- Cannot have an infinite sequence of reductions

Type checking

- Syntax of simply typed λ-calculus permits only well-typed terms
- Converse question; Given an arbitrary term, is it well-typed?
  - For instance, we cannot assign a valid type to $f\ f$ ...
  - ... so $f\ f$ is not a valid expression in this calculus

Theorem

The type-checking problem for the simply typed λ-calculus is decidable

Type checking ...

- A term may admit multiple types
  - $\lambda x.x$ can be of type $\tau \to \tau$, $(\tau \to \tau) \to (\tau \to \tau)$, ...
- Principal type scheme of a term M — unique type s such that every other valid type is an "instance" of s
  - Uniformly replace $\tau \in s$ by another type
  - $\tau \to \tau$ is principal type scheme of $\lambda x.x$

Theorem

We can always compute the principal type scheme for any well-typed term in the simply typed λ-calculus.

Computability with simple types

- Church numerals are well typed
- Translations of basic recursive functions (zero, successor, projection) are well-typed
- Translation of function composition is well typed
- Translation of primitive recursion is well typed
- Translation of minimalization requires elimination of recursive definitions
  - Uses untypable expressions of the form $f\ f$
- Minimalization introduces non terminating computations, but we have strong normalization!
- However, there do exist total recursive functions that are not primitive recursive — e.g. Ackermann's function

Polymorphism

- Simply typed λ-calculus has explicit types
- Languages like Haskell have polymorphic types
  - Compare `id ::  a -> a`
    with $\lambda x.x : \tau \to \tau$
- Second-order polymorhpic typed lambda calculus (System F)
  - Jean-Yves Girard
  - John Reynolds

System F

- Add type variables, a, b, ...
- Use i, j, ... to denote concrete types
- Type schemes

$$s ::= a \mid i \mid s \rightarrow s \mid \forall a.s$$

System F

Syntax of second order polymorphic lambda calculus

- Every variable and (type) constant is a term.
- If M is a term, x is a variable and s is a type scheme, then $(\lambda x \in s.M)$ is a term.
- If M and N are terms, so is (MN).
  - Function application does not enforce type check
- If M is a term and a is a type variable, then $(\Lambda a.M)$ is a term.
  - Type abstraction
- If M is a term and s is a type scheme, (Ms) is a term.
  - Type application

System F

Example A polymorphic identity function

$$\Lambda a.\lambda x \in a.x$$

Two β rules, for two types of abstraction

- $(\lambda x \in s.M)N \rightarrow_\beta M\{x \leftarrow N\}$
- $(\Lambda a.M)s \rightarrow_\beta M\{a \leftarrow s\}$