Programming Language Concepts: Lecture 22

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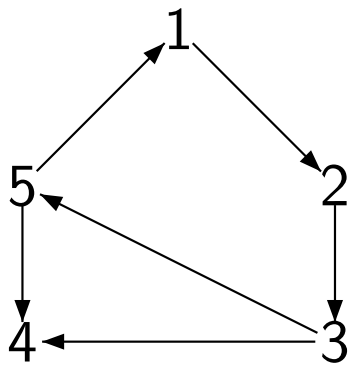
Logic programming

- Programming with relations
- Variables
  - Names starting with a capital letter
  - `X`, `Y`, `Name`, ...
- Constants
  - Names starting with a small letter
  - `ball`, `node`, `graph`, `a`, `b`, ...
  - Uninterpreted — no types like `Char`, `Bool` etc!
  - Exception: natural numbers, some arithmetic

Defining relations

A Prolog program describes a relation

Example: A graph



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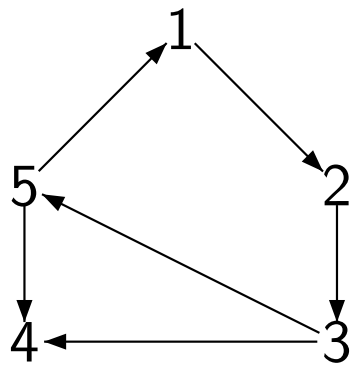
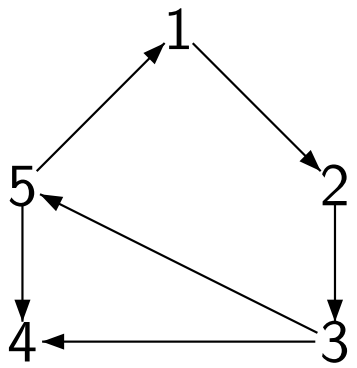


- Want to define a relation `path(X,Y)`
- `path(X,Y)` holds if there is a path from `X` to `Y`

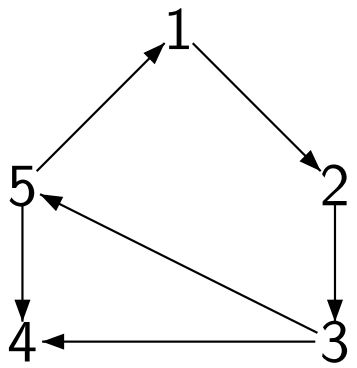
Facts and rules



Represent edge relation using the following facts.

```
edge(3,4).
edge(5,4).
edge(5,1).
edge(1,2).
edge(3,5).
edge(2,3).
```

Facts and rules . . .



Define `path` using the following rules.

```
path(X,Y) :- X = Y.
path(X,Y) :- edge(X,Z), path(Z,Y).
```

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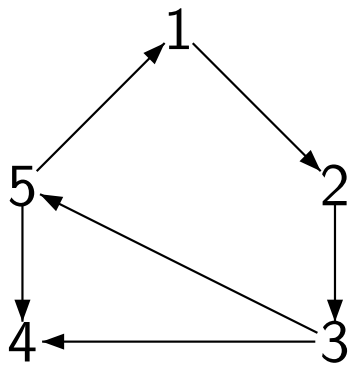

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```

Read the rules read as follows:

Rule 1 For all `X`, `(X,X)` $\in$ `path`.

Rule 2 For all `X`,`Y`, `(X,Y)` $\in$ `path` if there exists `Z` such that `(X,Z)` $\in$ `edge` and `(Z,Y)` $\in$ `path`.

Facts and rules . . .

```
path(X,Y) :- X = Y.
path(X,Y) :- edge(X,Z), path(Z,Y).
```

- Each rule is of the form

  Conclusion if Premise_1 and Premise_2 . . . and Premise_n

  - if is written `:-`
  - and is written `,`
  - This type of logical formula is called a Horn Clause

- Quantification of variables
  - Variables in goal are universally quantified
    - `X`, `Y` above
  - Variables in premise are existentially quantified
    - `Z` above

Computing in Prolog

- Ask a question (a query)

```
?- path(3,1).
```

- Prolog scans facts and rules top-to-bottom
  - 3 cannot be unified with 1, skip Rule 1.
  - Rule 2 generates two subgoals. Find Z such that
    - (3,Z) ∈ edge and
    - (Z,1) ∈ path.
- Sub goals are tried depth-first
  - (3,Z) ∈ edge?
    - (3,4) ∈ edge, set Z = 4
  - (4,1) ∈ path? 4 cannot be unifed with 1, two subgoals, new Z'
    - (4,Z') ∈ edge
    - (Z',1) ∈ path
  - Cannot find Z' such that (4,Z') ∈ edge!

Backtracking

- `(3,Z)` $\in$ `edge`?
  - `edge(3,4)` $\in$ `edge`, set `Z = 4`
- `(4,1)` $\in$ `path`? `4` cannot be unified with `1`, two subgoals, new `Z'`
  - `(4,Z')` $\in$ `edge`
  - `(Z',1)` $\in$ `path`
- No `Z'` such that `(4,Z')` $\in$ `edge`
- Backtrack and try another value for `Z`
  - `edge(3,5)` $\in$ `edge`, set `Z = 5`
- `(5,1)` $\in$ `path`? `(5,1)` $\in$ `edge`, $\surd$

Backtracking is sensitive to order of facts

- We had put `edge(3,4)` before `edge(3,5)`

Reversing the question

- Consider the question

```
?- edge(3,X).
```

- Find all `X` such that `(3,X)` $\in$ `edge`
- Prolog lists out all satisfying values, one by one

```
X=4;
X=5;
X=2;
No.
```

Unification and pattern matching

- A goal of the form `X = Y` denotes unification.

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```

- Unification provides a formal justification for pattern matching in rule definitions
  - Unlike Haskell, a repeated variable in the pattern is meaningful
  - In Haskell, we cannot write

```
path (x,x) = True
```

Complex data and terms

Represent arbitrary structures with nested terms

- A record or struct of the form

```
personal_data{
   name : amit
   date_of_birth{
      year  : 1980
      month : 5
      day   : 30
   }
}
```

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```

- . . . can be represented by a term

```
personal_data(name(amit),
              date_of_birth(year(1980),month(5),day(30)))
```

Lists

- Write `[Head | Tail]` for Haskell's `(head:tail)`

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- Checking membership in a list

```
member(X,[Y|T]) :- X = Y.
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```

- . . . plus anonymous variables.

```
member(X,[X|_]).
member(X,[_|T]) :- member(X,T).
```

Lists . . .

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- Inductive definition, like Haskell

```
append(Xs, Ys, Zs) :- Xs = [], Zs = Ys.
append(Xs, Ys, Zs) :- Xs = [H | Ts], Zs = [H | Us],
                      append(Ts, Ys, Us).
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```

- Again, eliminate explicit unification

```
append([], Ys, Ys).
append([X | Xs], Ys, [X | Zs]) :- append(Xs, Ys, Zs).
```

Reversing the computation

```
?- append(Xs, Ys, [mon, wed, fri]).
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All possible ways to split the list

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All possible ways to split the list

```
Xs = []
Ys = [mon, wed, fri] ;

Xs = [mon]
Ys = [wed, fri] ;

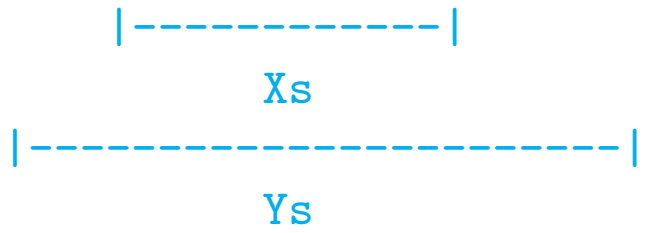
Xs = [mon, wed]
Ys = [fri] ;

Xs = [mon, wed, fri]
Ys = [] ;

no
```

Reversing the computation . . .

- Want to define a relation `sublist(Xs,Ys)`



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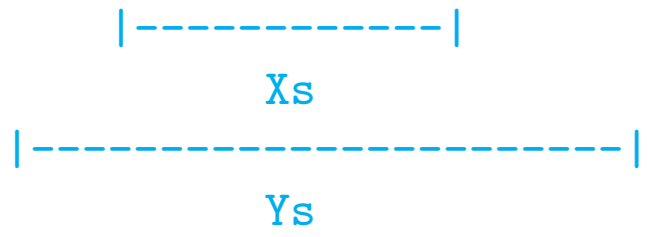

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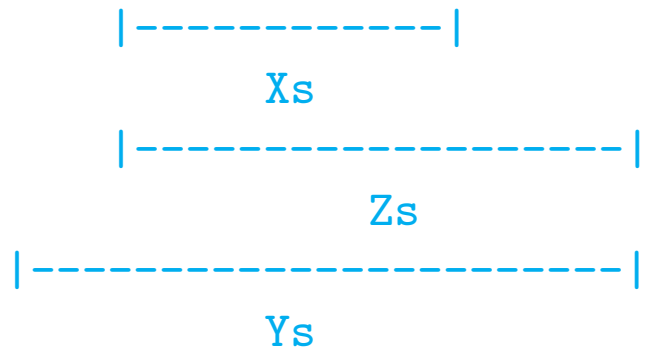

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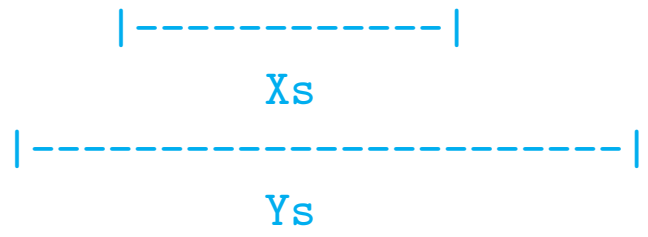

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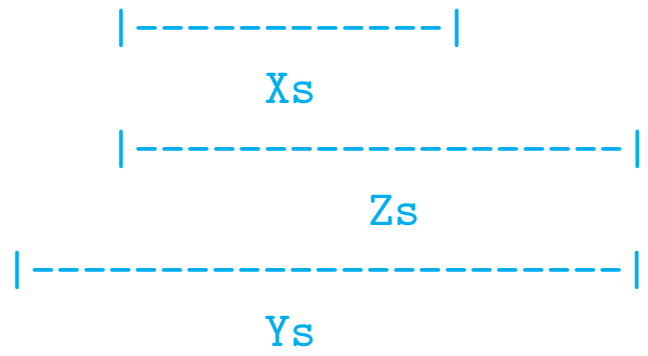

- Yields the rule

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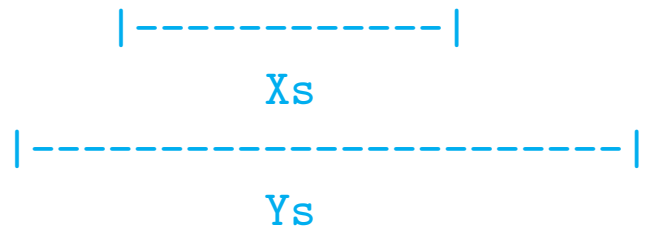

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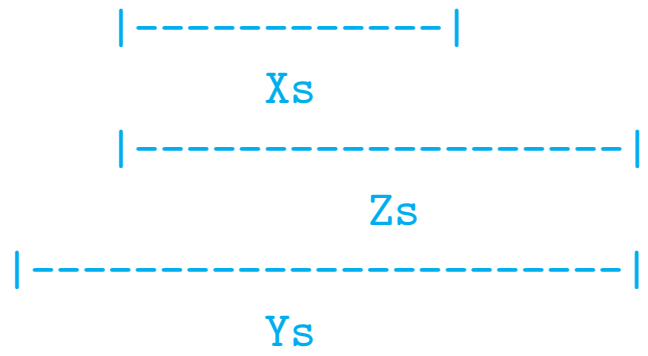

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```

- Why won't the following work?

```
sublist(Xs, Ys) :- append(Xs, _, Zs), append(_, Zs, Ys).
```

Reversing the computation . . .

Type inference for simply typed lambda calculus

$$x \in \mathit{Var} \mid \lambda x.M \mid MN$$

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- Inference rules to derive type judgments of the form $A \vdash M : s$
  - A is list $\{x_i : t_i\}$ of type "assumptions" for variables
  - Under the assumptions in `A` the expression M has type s.

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$$\frac{x : t \in A}{A \vdash x : t}$$

$$\frac{A \vdash M : s \to t, \quad A \vdash N : s}{A \vdash (MN) : t}$$

$$\frac{A + x : s \vdash M : t}{A \vdash (\lambda x.M) : s \to t}$$

Reversing the computation . . .

- Encoding λ-calculus and types in Prolog
  - `var(x)` for variable x (Note: `x` is a constant!)
  - `lambda(x,m)` for $\lambda x.M$
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member([x, arrow(S,T)], [[x, S]])
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member([x, arrow(S,T)], [[x, S]])
```

  - Unification fails

Example: special sequence . . .

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$$1, 9, 1, 2, 1, 8, 2, 4, 6, 2, 7, 9, 4, 5, 8, 6, 3, 4, 7, 5, 3, 9, 6, 8, 3, 5, 7.$$

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```
% sequence(Xs) :- Xs is a list of 27 variables.
sequence([_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_]).
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```
% sequence(Xs) :- Xs is a list of 27 variables.
sequence([_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_,_]).

solution(Ss) :-
sequence(Ss),
sublist([9,_,_,_,_,_,_,_,_,9,_,_,_,_,_,_,_,_,9], Ss),
sublist([8,_,_,_,_,_,_,_,8,_,_,_,_,_,_,_,8], Ss),
sublist([7,_,_,_,_,_,_,7,_,_,_,_,_,_,7], Ss),
sublist([6,_,_,_,_,_,6,_,_,_,_,_,6], Ss),
sublist([5,_,_,_,_,5,_,_,_,_,5], Ss),
sublist([4,_,_,_,4,_,_,_,4], Ss),
sublist([3,_,_,3,_,_,3], Ss),
sublist([2,_,2,_,2], Ss),
sublist([1,1,1], Ss).
```

Arithmetic

Computing length of a list

```
length([],0).
length([H,T],N) :- length(T,M), N = M+1.
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```
N=0+1+1+1+1
```

- `X = Y` is unification

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What does the following query yield?

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?- length([1,2,3,4],N).
```

```
N=0+1+1+1+1
```

- `X = Y` is unification
- `X is Y` captures arithmetic equality

Arithmetic

Computing length of a list

```
length([],0).
length([H,T],N) :- length(T,M), N = M+1.
```

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length([],0).
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```

Arithmetic . . .

Another approach

```
length(L,N) :- auxlength(L,0,N).
auxlength([],N,N).
auxlength([H|T],M,N) :- auxlength(T,M1,N), M1 is M+1.
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Arithmetic . . .

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```

`?- length([0,1,2],N)` generates goals

```
auxlength([0,1,2],0,N)
```

Arithmetic . . .

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`?- length([0,1,2],N)` generates goals

`auxlength([0,1,2],0,N)`

`auxlength([1,2],1,N)`

Arithmetic . . .

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`?- length([0,1,2],N)` generates goals

```
auxlength([0,1,2],0,N)
auxlength([1,2],1,N)
auxlength([2],2,N)
```

Arithmetic . . .

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Arithmetic . . .

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`?- length([0,1,2],N)` generates goals

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auxlength([],3,N)
auxlength([],3,3)
```

Arithmetic . . .

Another approach

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length(L,N) :- auxlength(L,0,N).
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`?- length([0,1,2],N)` generates goals

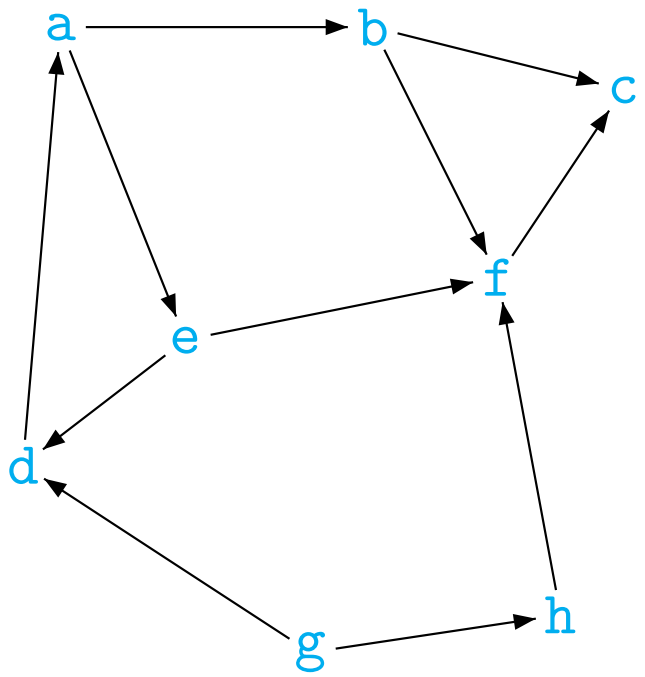
```
auxlength([0,1,2],0,N)
auxlength([1,2],1,N)
auxlength([2],2,N)
auxlength([],3,N)
auxlength([],3,3)
```

Second argument to `auxlength` accumulates answer.

Coping with circular definitions

```
edge(g,h).
edge(d,a).
edge(g,d).
edge(e,d).
edge(h,f).
edge(e,f).
edge(a,e).
edge(a,b).
edge(b,f).
edge(b,c).
edge(f,c).

path(X,X).
path(X,Y) :- edge(X,Z), path(Z,Y).
```



Coping with circular definitions

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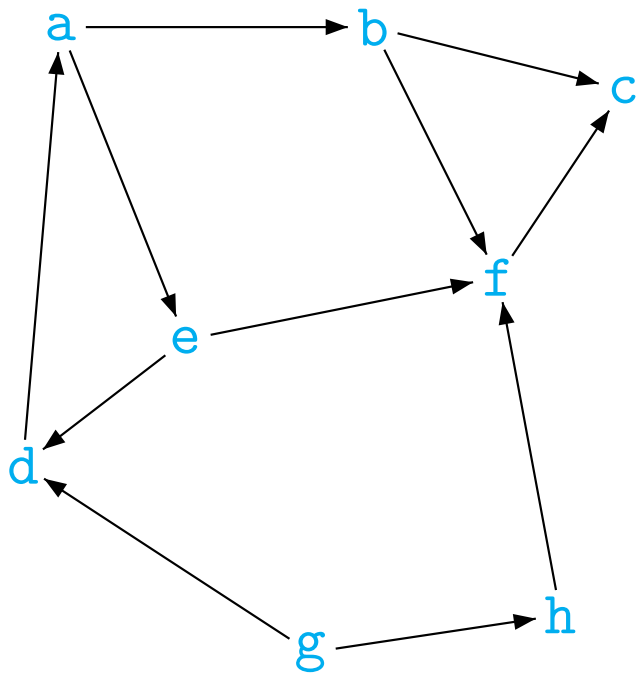

What does `?- path(a,b)` compute?

Coping with circularity . . .

Instead

```
path(X,X,T).
path(X,Y,T) :- a(X,Z), legal(Z,T), path(Z,Y,[Z|T]).

legal(Z,[]).
legal(Z,[H|T]) :- Z\==H, legal(Z,T).
```

- `path(X,Y,T)` succeeds if there is a path from `X` to `Y` that does not visit any nodes in `T`
- `T` is an accumulator