Lecture 17: 19 March, 2024

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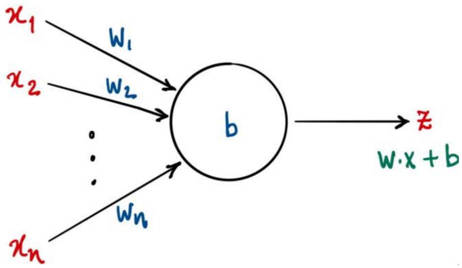
Data Mining and Machine Learning
January–April 2024

Linear separators and Perceptrons

- Perceptrons define linear separators $w \cdot x + b$
  - $w \cdot x + b > 0$, classify Yes $(+1)$
  - $w \cdot x + b < 0$, classify No (-1)



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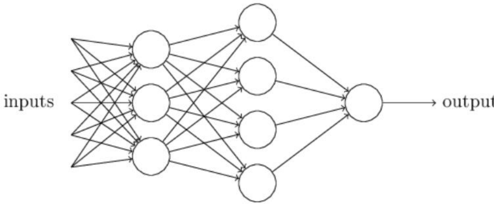

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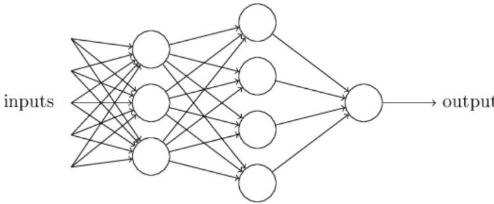

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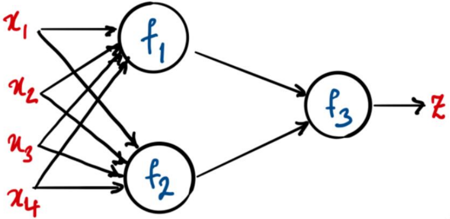

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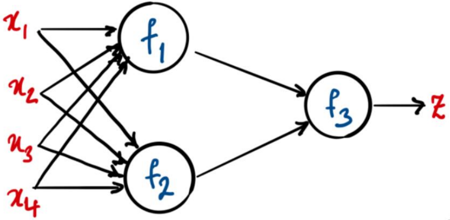

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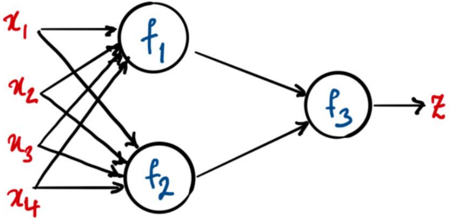

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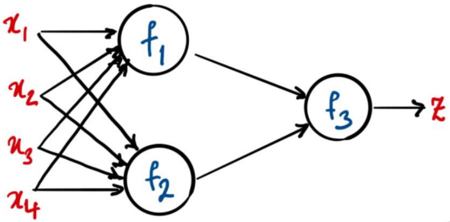
  - $f_3 = \sum_{i=1}^{4} (w_{3_1} w_{1_i} + w_{3_2} w_{2_i}) \cdot x_i$ $+(w_{3_1} b_1 + w_{3_2} b_2 + b_3)$



Limits of linearity

- Cannot compute *exclusive-or* (XOR)
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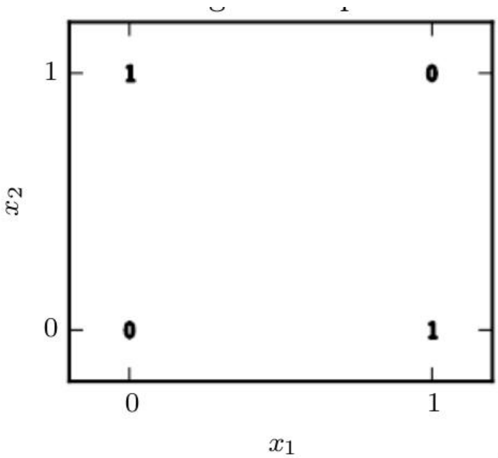

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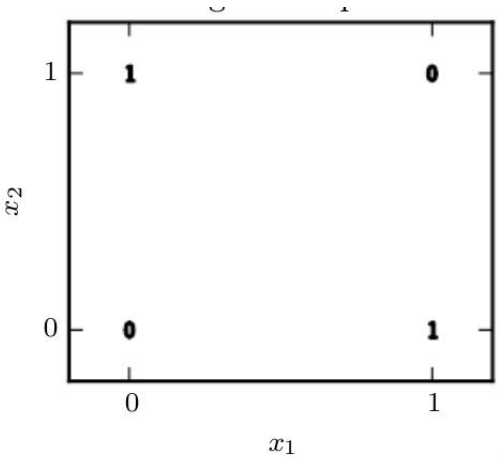

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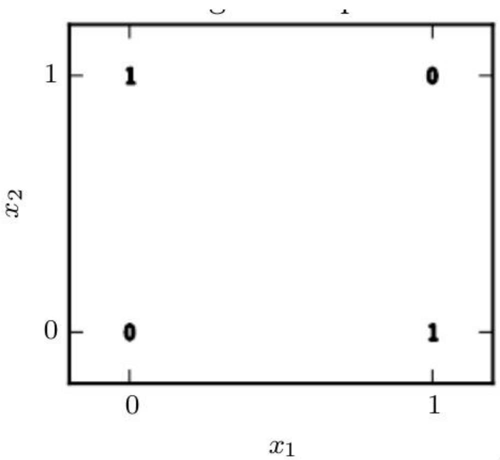

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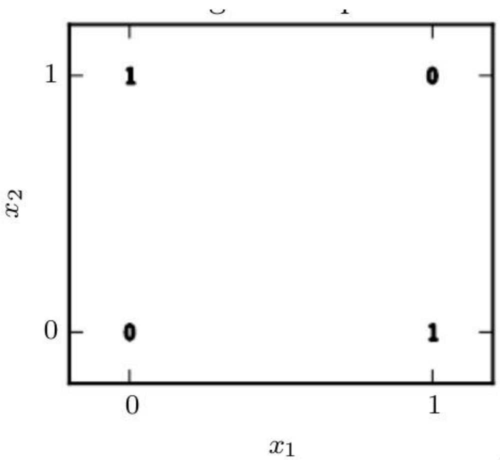

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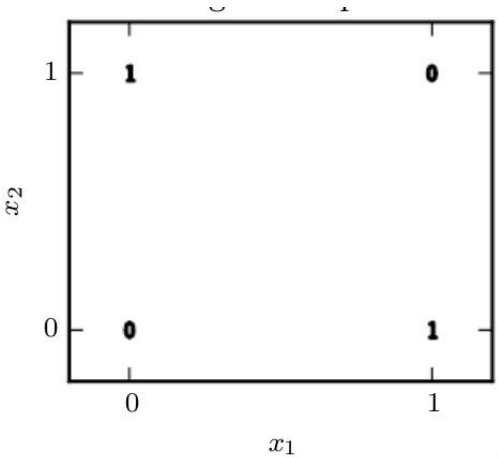
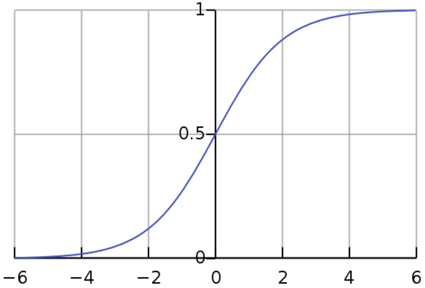
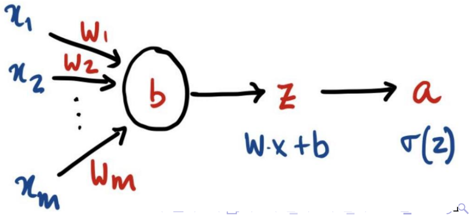
- Observed by Minsky and Papert, 1969, first "AI Winter"



Non-linear activation

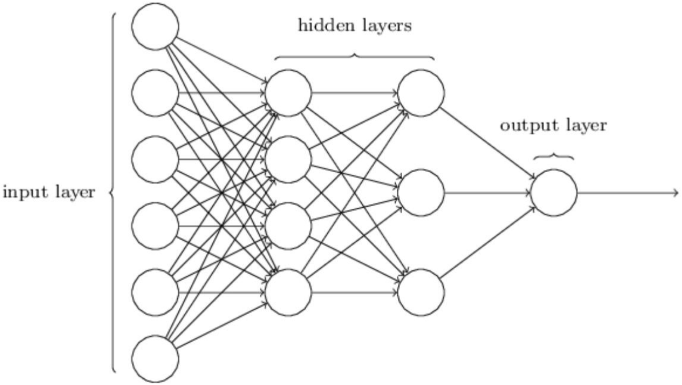
- Transform linear output z through a non-linear activation function
- Sigmoid function $\dfrac{1}{1+e^{-z}}$





Structure of a neural network

- Acyclic
- Input layer, hidden layers, output layer



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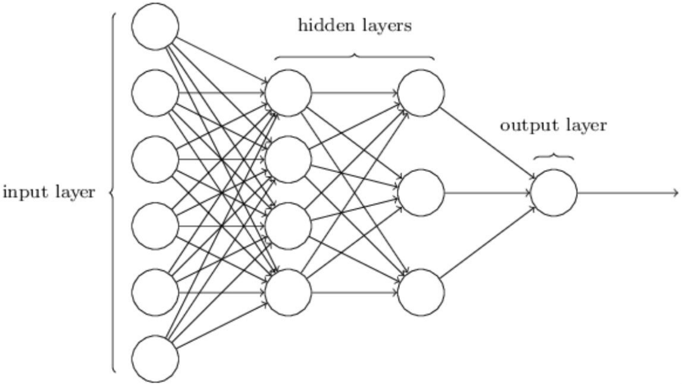

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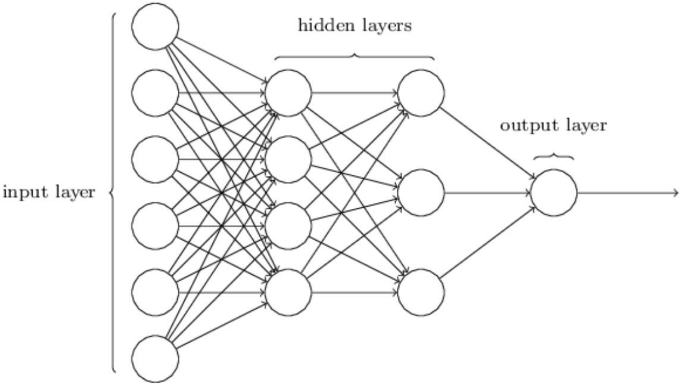

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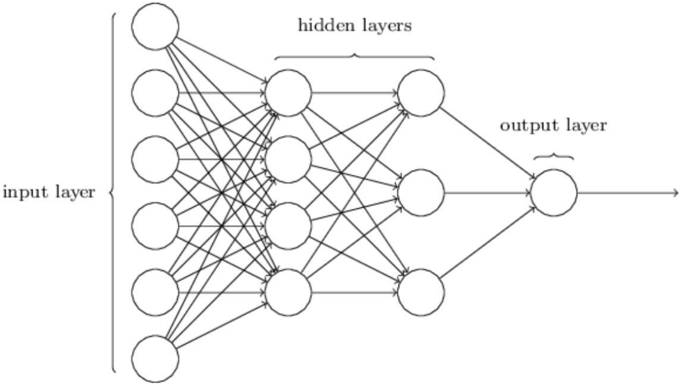

Structure of a neural network

- Acyclic
- Input layer, hidden layers, output layer
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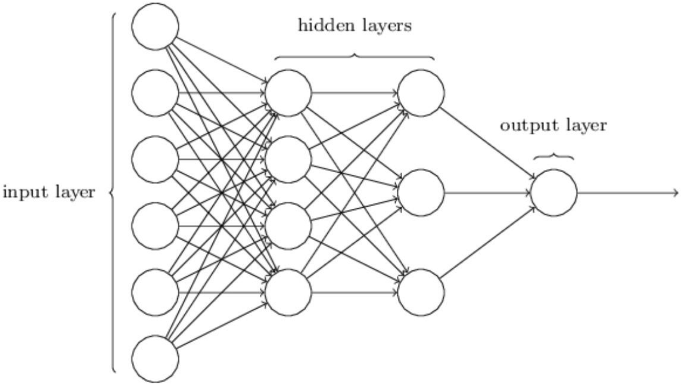
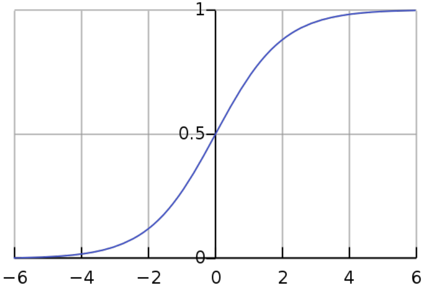
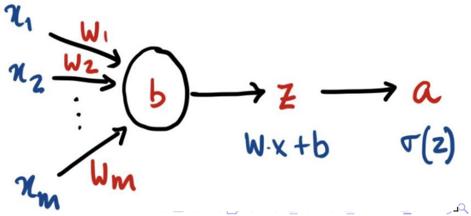
  - Hidden neurons are arranged in layers
  - Each layer is fully connected to the next
  - Set weight to zero to remove an edge



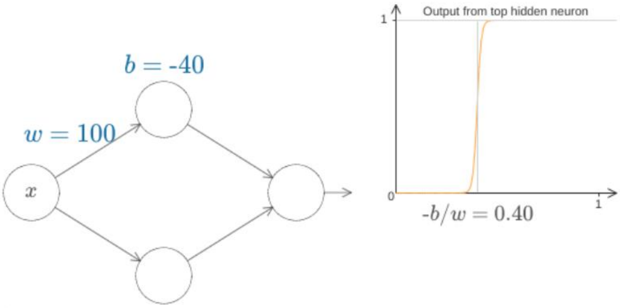
Non-linear activation

- Transform linear output z through a non-linear activation function
- Sigmoid function $\dfrac{1}{1+e^{-z}}$
- Step is at $z = 0$
  - $z = wx + b$, so step is at $x = -b/w$
  - Increasing w makes step steeper

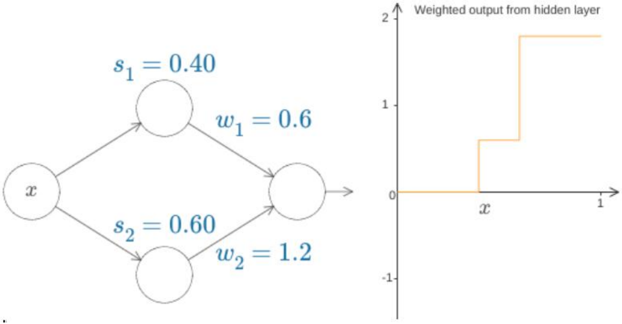




- Create a step at $x = -b/w$

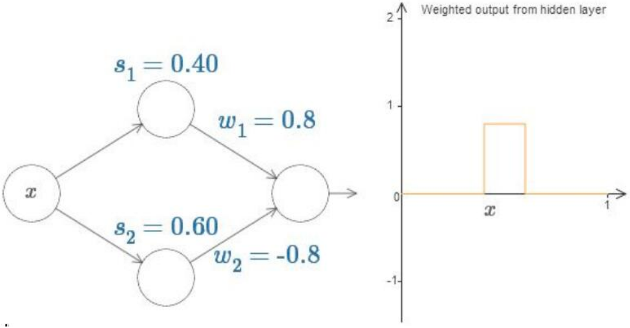


- Create a step at $x = -b/w$
- Cascade steps



Universality

- Create a step at $x = -b/w$
- Cascade steps
- Subtract steps to create a box



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- Create a step at $x = -b/w$
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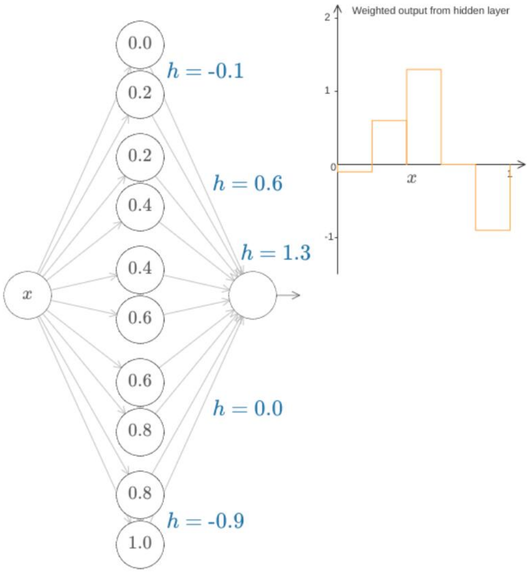
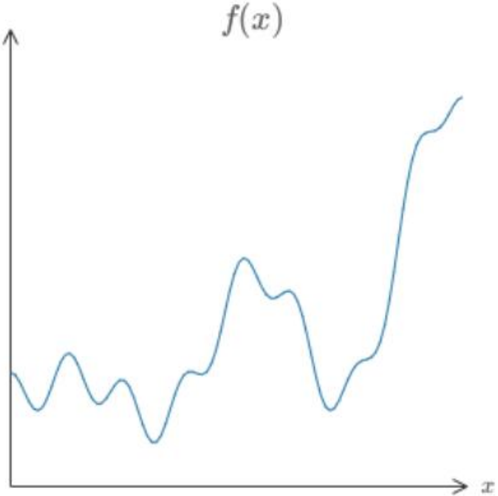


- Create a step at $x = -b/w$
- Cascade steps
- Subtract steps to create a box
- Create many boxes
- Approximate any function



Universality

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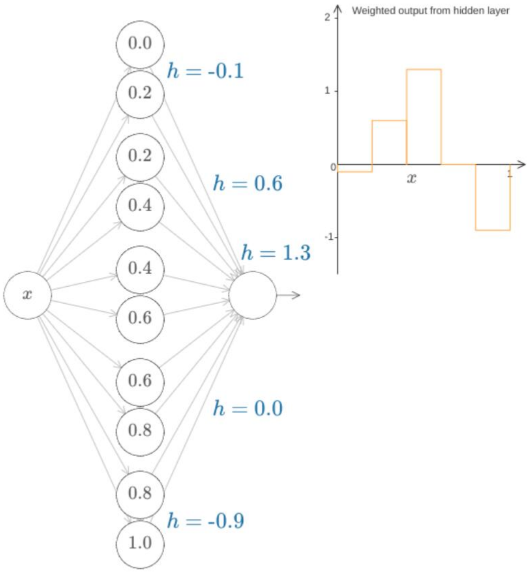
- Create many boxes
- Approximate any function
- Need only one hidden layer!



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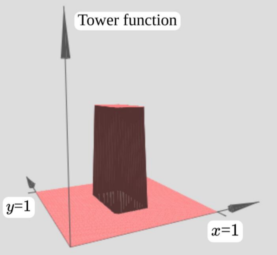
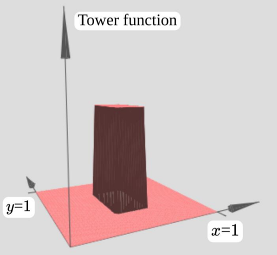
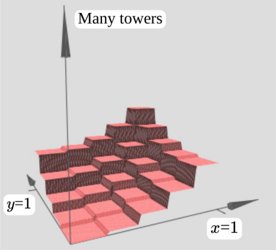


Non-linear activation

- With non-linear activation, network of neurons can approximate any function
  - Can build “rectangular” blocks
  - Combine blocks to capture any classification boundary





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- 1000 samples of 10 handwritten digits
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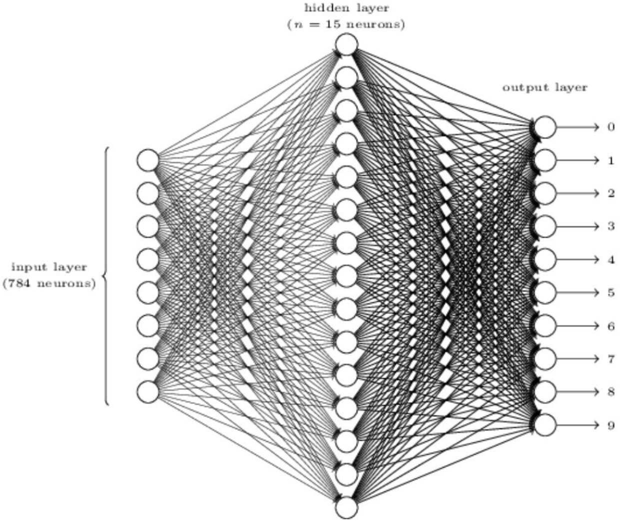
- Input $x = (x_1, x_2, \ldots, x_{784})$

Example: Network structure

- Input layer $(x_1, x_2, \ldots, x_{784})$



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- Input layer $(x_1, x_2, \ldots, x_{784})$
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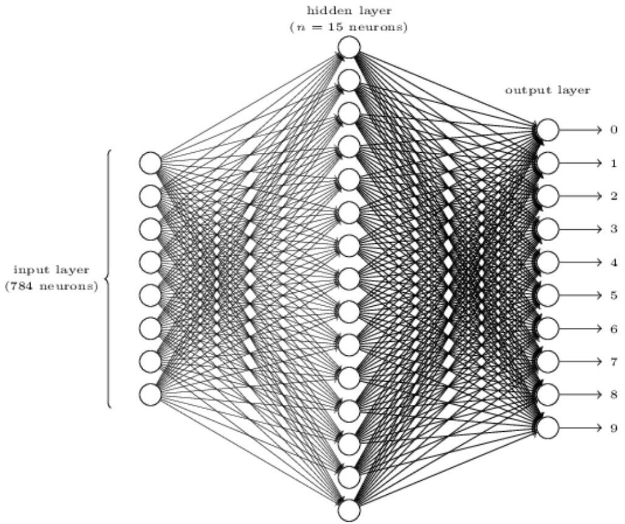

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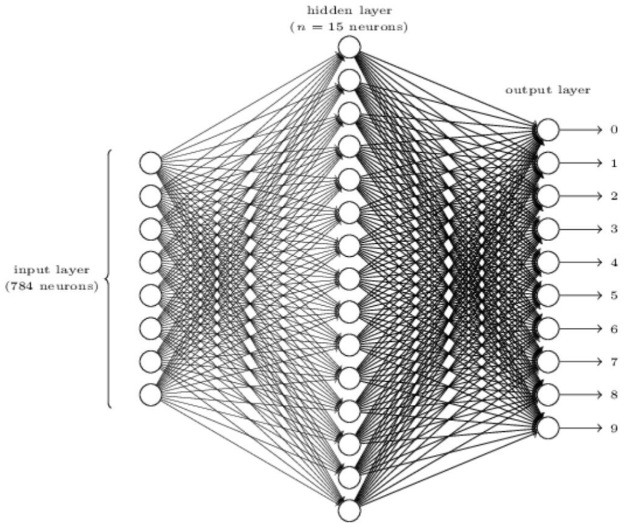
  - Decision a_j for each digit $j \in \{0, 1, \ldots, 9\}$



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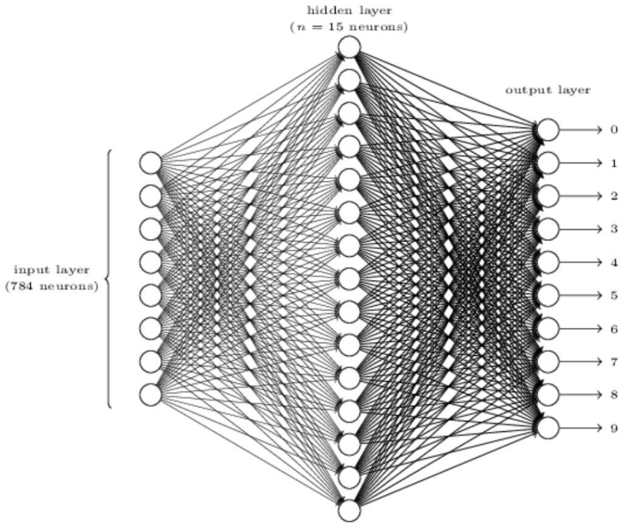

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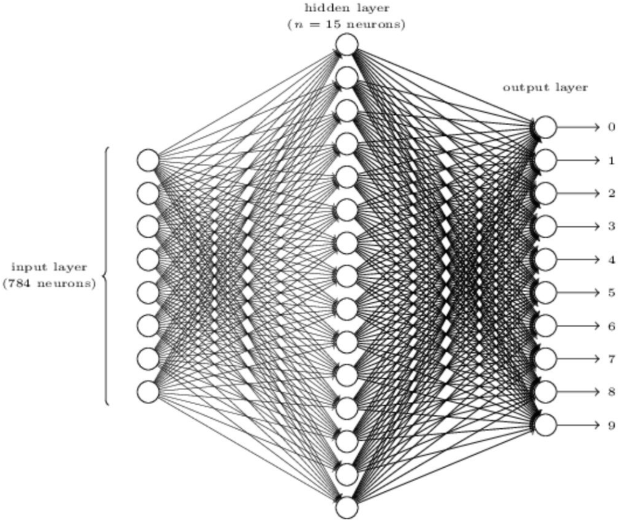

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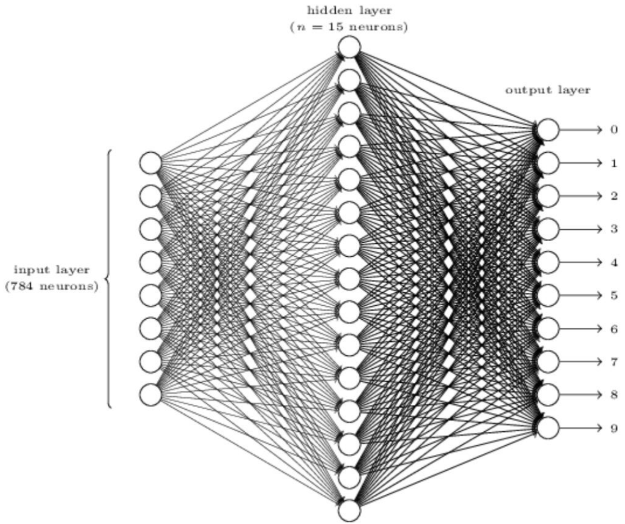
  - Softmax, $\arg\max_j \dfrac{e^{a_j}}{\sum_j e^{a_j}}$
    - "Smooth" version of $\arg\max$



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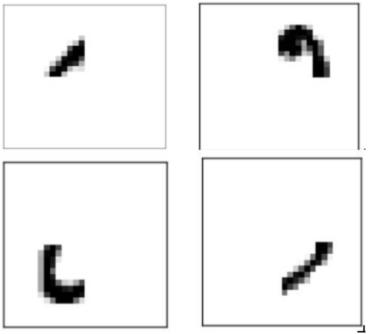
- Hidden layers extract features
  - For instance, patterns in different quadrants



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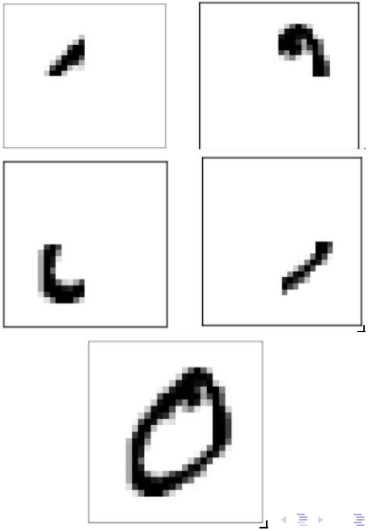

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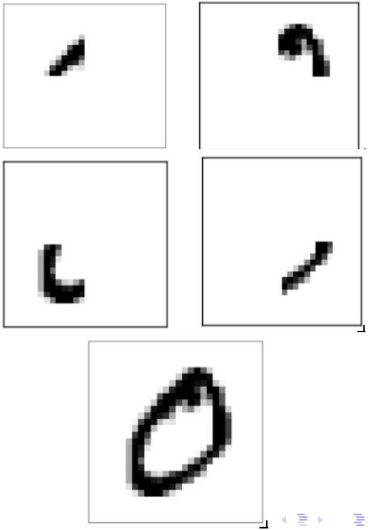

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- Hidden layers extract features
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- Claim: Automatic identification of features is strength of the model
- Counter argument: implicitly extracted features are impossible to interpret
  - Explainability