Lecture 20: 28 March, 2024

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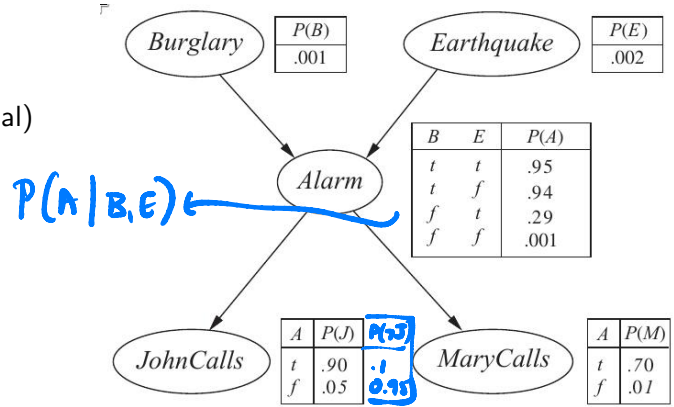
Data Mining and Machine Learning
January–April 2024

Probabilistic graphical models

- Underlying DAG, no cyclic dependencies
- Each node has a local (conditional) probability table



John calls, Mary calls — was there a burglary?

$$P(B \mid J, M) = x \frac{P(B, J, M)}{P(J, M)}$$

$$P(\neg B \mid J, M) = y \frac{P(\neg B, J, M)}{P(J, M)}$$

$+ = 1$

$x + y = 1 \qquad \frac{x}{y} = \frac{P(B, J, M)}{P(\neg B, J, M)}$

$$\frac{P(B, J, M)}{\alpha} = P(B \mid J, M)$$

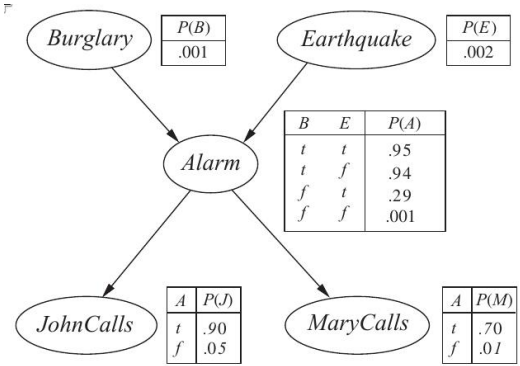
$$P(\neg B, J, M) = P(\neg B \mid J, M) \cdot \alpha$$

$$P(B, J, M) + P(\neg B, J, M) = \left(P(B \mid J, M) + P(\neg B \mid J, M)\right) \alpha$$

$$P(J, M) = \sum_{B=0,1} P(B, J, M)$$

Conditional independence

- $x \perp y$ — x and y are independent
  - $P(x \wedge y) = P(x) \cdot P(y)$



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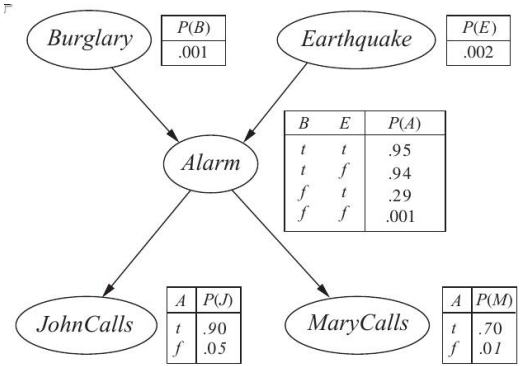
$P(x \mid y, z) = P(x \mid z)$



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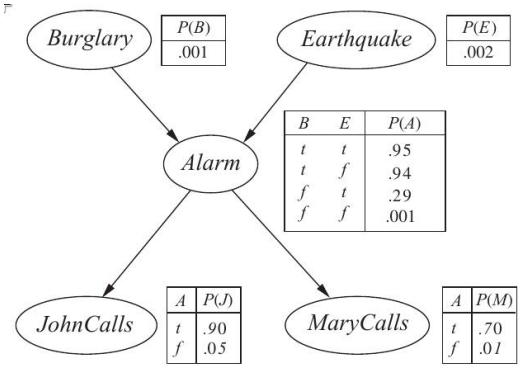

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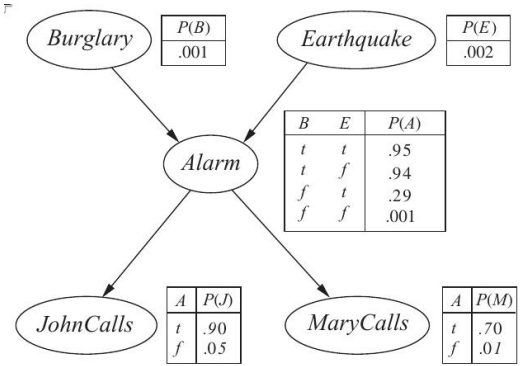

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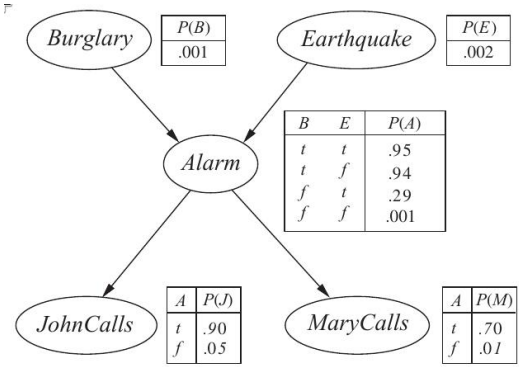

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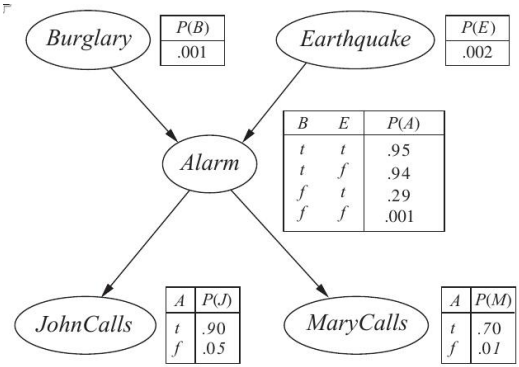
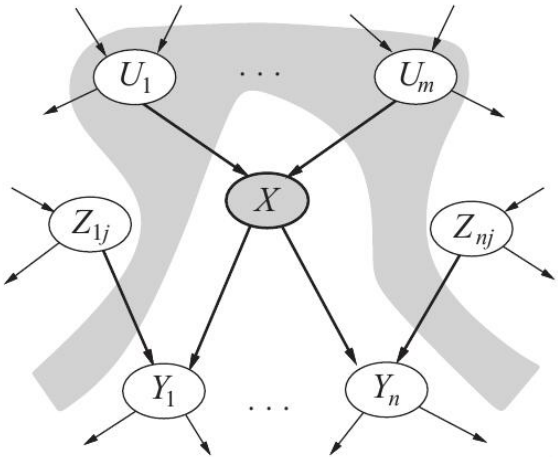
  - Yes — by semantics of network, local independence



Probabilistic graphical models

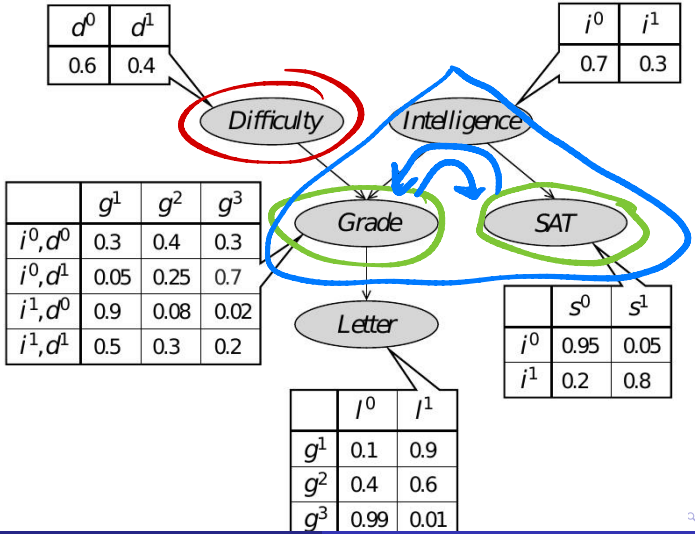
- Fundamental assumption

  A node is conditionally independent of non-descendants, given its parents



Student example

- $SAT \perp Grade \mid Difficulty$?



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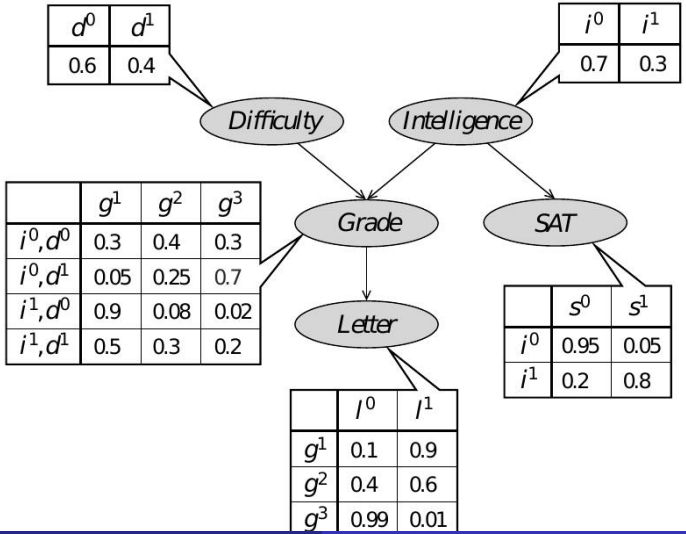

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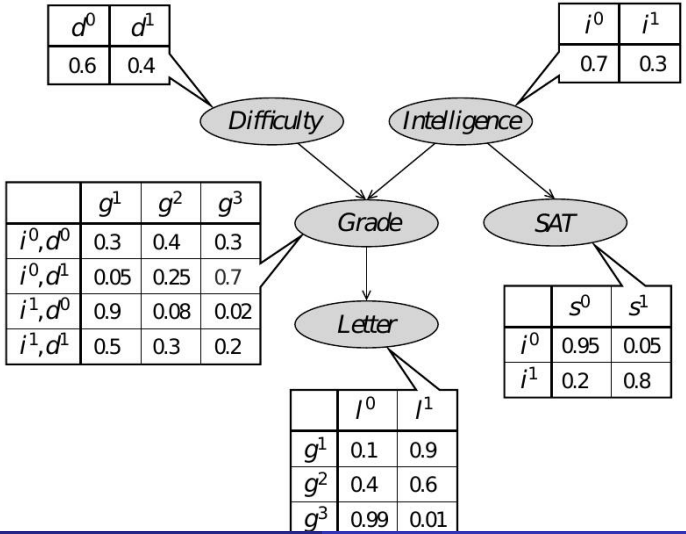
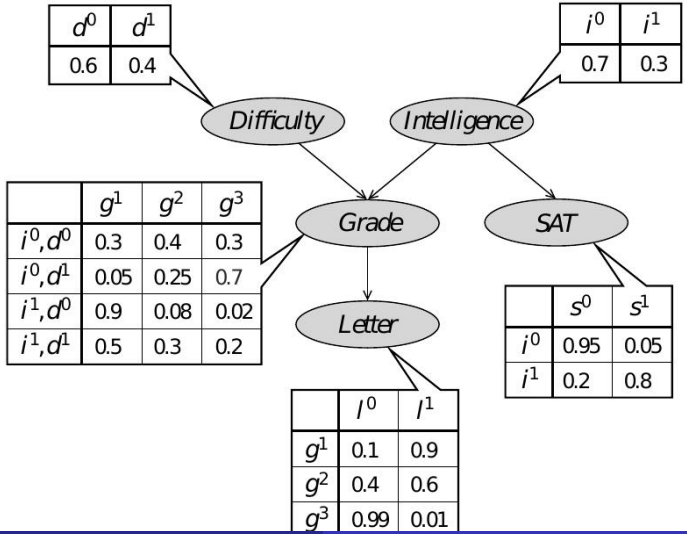


Student example

- $SAT \perp Grade \mid Difficulty$?
  - No
- Can we calculate conditional independence from the graph?
- In general, check if $X \perp Y \mid Z$ for sets of variables X, Y, Z



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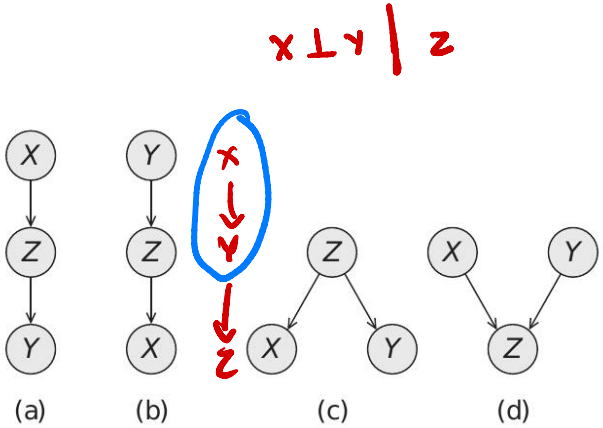
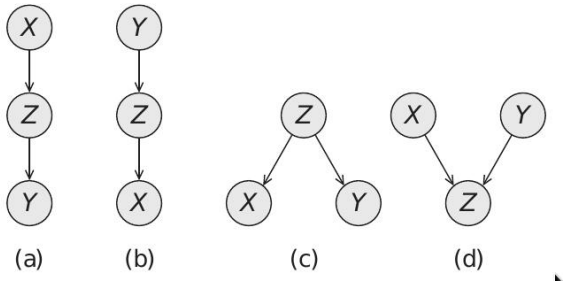
- Examine trails between nodes
  - Paths in the underlying undirected graph
- Basic trails — (undirected) paths of length 2
  - Four basic trails



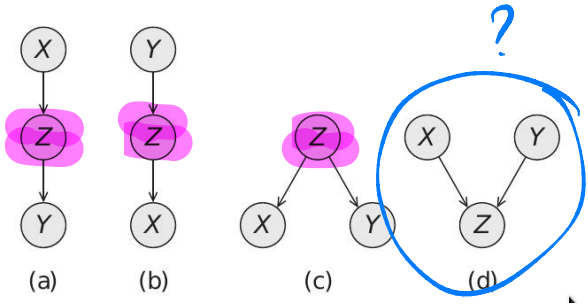
Basic trails



Basic trails

- (a), (b) and (c): Z blocks flow between X and Y, by semantics of Bayesian networks

$$P(X \wedge Y \mid Z) = P(Y \wedge X \mid Z)$$



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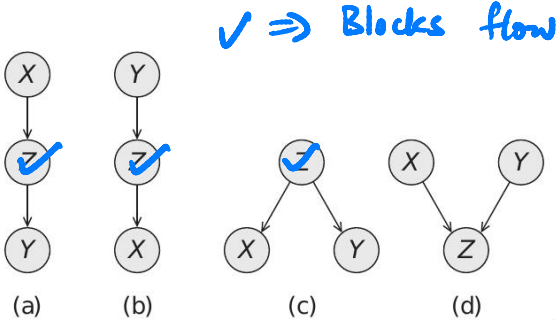

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  - Z: Car does not start
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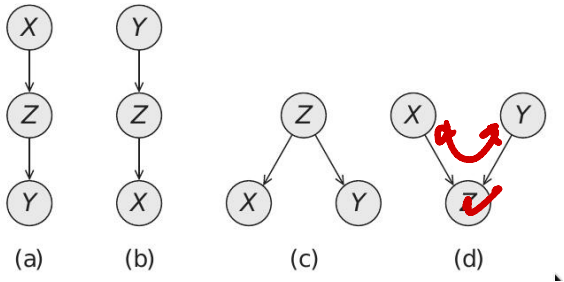

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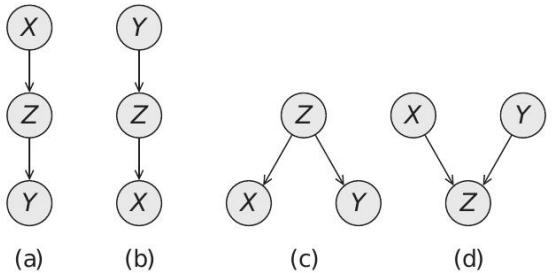

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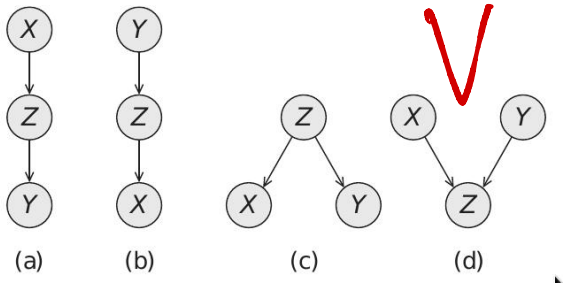
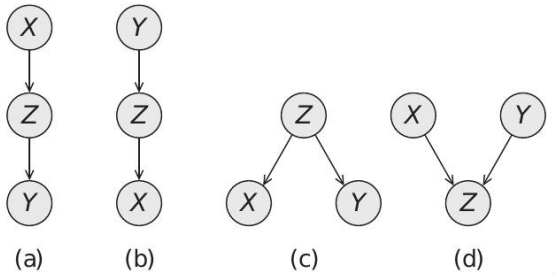
  - Simplest form of V-structure



D-Separation

- Check if $X \perp Y \mid Z$



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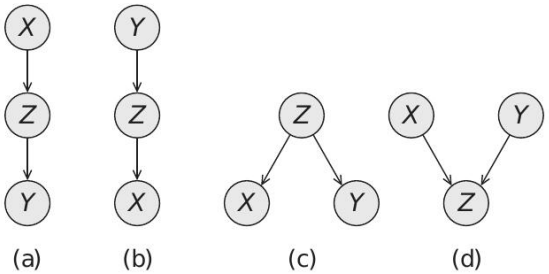
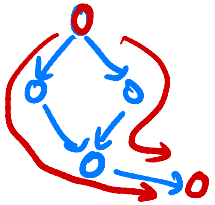
- Check if $X \perp Y \mid Z$
- Dependence should be blocked on every trail from X to Y





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- Check if $X \perp Y \mid Z$
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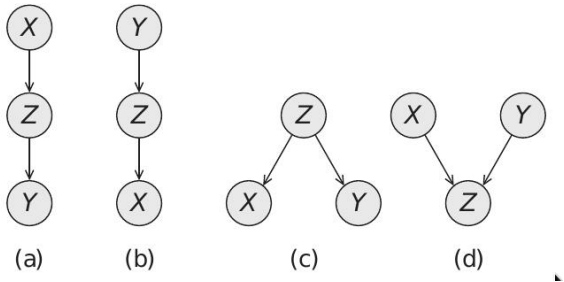

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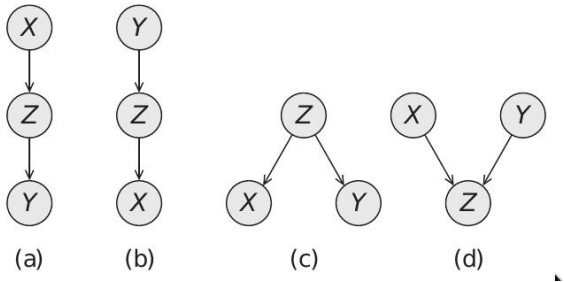

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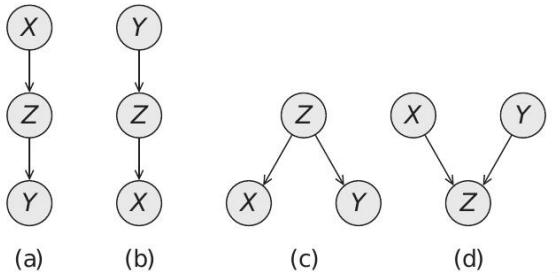

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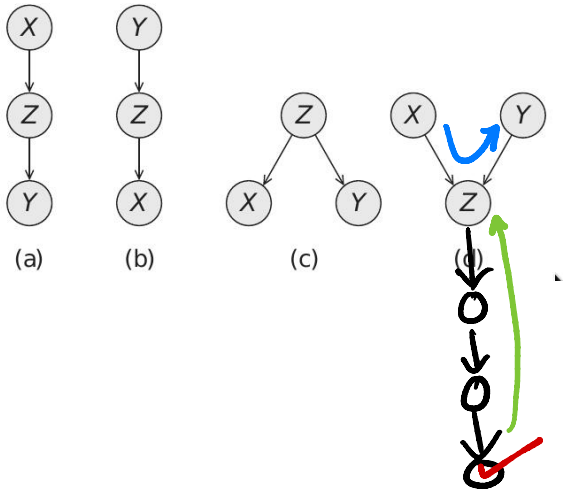
  - In general, V-structure includes descendants of the bottom node



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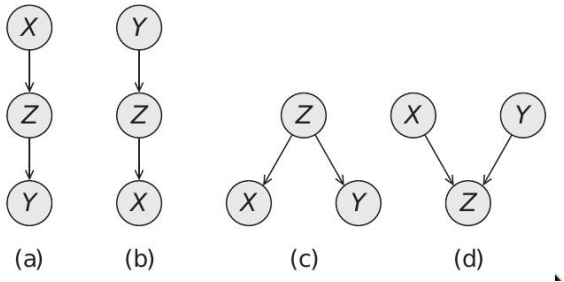

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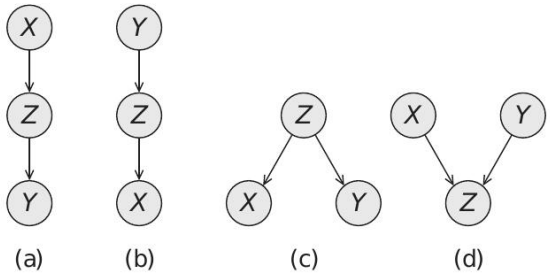
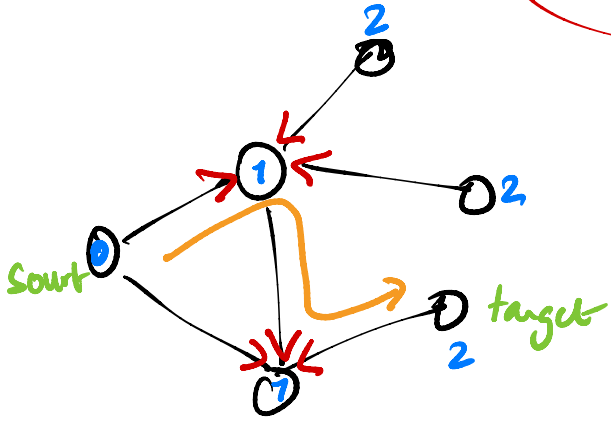


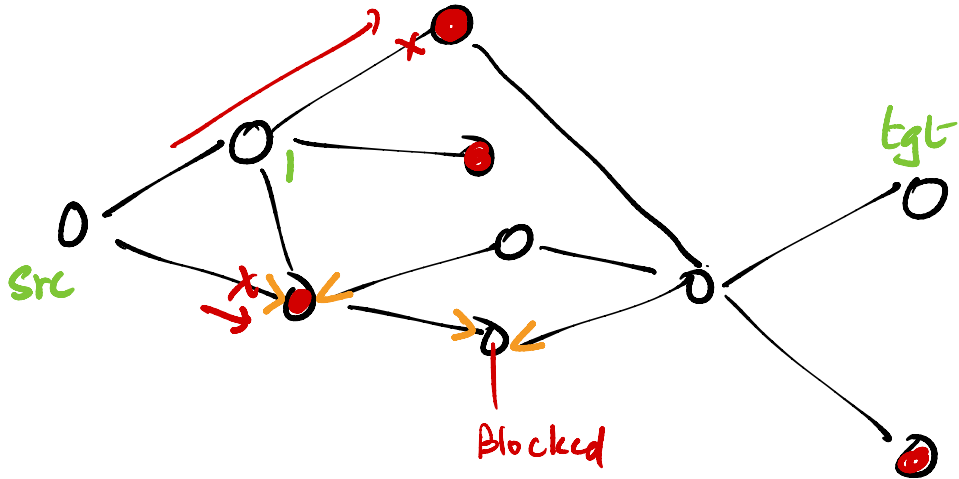
- x and y are D-separated given z if all trails are blocked
- Variation of breadth first search (BFS) to check if y is reachable from x through some trail

Breadth first search

Drop >

?

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Blocked

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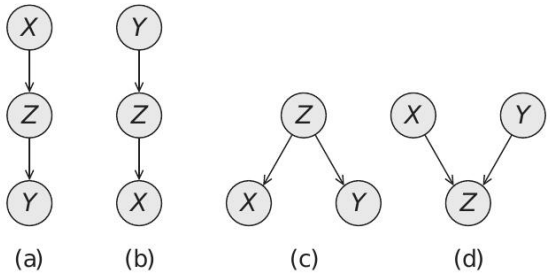
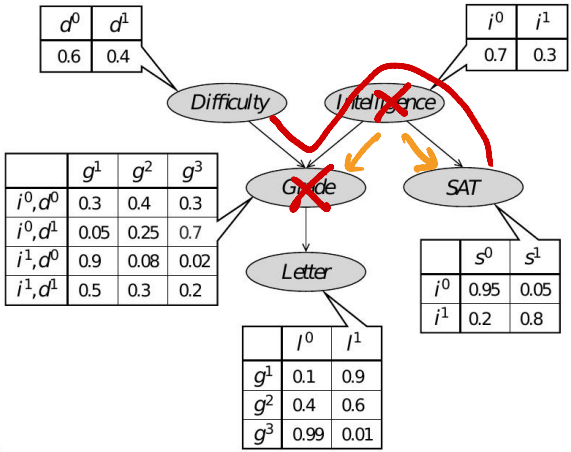


- x and y are D-separated given z if all trails are blocked
- Variation of breadth first search (BFS) to check if y is reachable from x through some trail
- Extends to sets — each $x \in X$ is D-separated from each $y \in Y$

$|X| \approx |Y|$

Conditional independence, example

- Is SAT independent of Difficulty given Intelligence?
  - Yes, Difficulty – Grade – Intelligence – SAT trail is blocked at Grade (V-structure) and Intelligence



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- Is SAT independent of Difficulty given Intelligence?
  - Yes, Difficulty – Grade – Intelligence – SAT trail is blocked at Grade (V-structure) and Intelligence
- Is SAT independent of Difficulty given Letter?
  - No, Difficulty – Grade – Intelligence – SAT trail is open
  - Letter is known, hence something about Grade is known (V-structure)
  - Intelligence is not known