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Data Mining and Machine Learning
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Markov Decision Process

S – states

A – actions

$$\begin{tikzcd} S_t \arrow[r] & A_t \arrow[r] \arrow[dr] & S_{t+1} \\ & \text{agent chooses} \arrow[u] & R_{t+1} \end{tikzcd}$$

$$p(s', r \mid s, a)$$

Policy π Given s, choose a

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V_π

V_* is

V_π at π_*

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- Bellman optimality equation for v_*

$$v_*(s) = \max_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma v_*(s')]$$

not a linear operator,

Not LP

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- n large, computationally infeasible — use iterative methods to approximate v_*

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  - Stop when incremental change $\Delta = |v_\pi^{k+1} - v_\pi^k|$ is below threshold θ

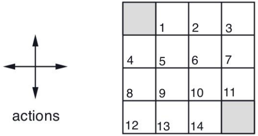
Policy evaluation

Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Policy evaluation example



$R_t = -1$
on all transitions

Moves deterministic

Hitting boundary – no change of position

v_k for the random policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

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$$= \sum_{s', r} p(s', r \mid s, a) \left[r + \gamma v_\pi(s')\right]$$

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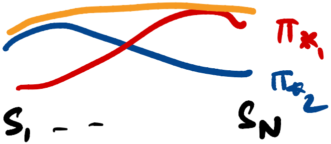

- If $q_\pi(s,a) > v_\pi(s)$, modify π so that $\pi(s) = a$

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- $$\begin{aligned} q_\pi(s,a) &= \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \sum_{s',r} p(s',r \mid s,a)\left[r + \gamma v_\pi(s')\right] \end{aligned}$$
- If $q_\pi(s,a) > v_\pi(s)$, modify π so that $\pi(s) = a$
- The new policy π' is strictly better

Policy improvement

Policy Improvement Theorem

For deterministic policies π, π':

- If $q_\pi(s, \pi'(s)) \geq v_\pi(s)$ for all s, then $\pi' \geq \pi$,
- If $\pi' \geq \pi$ and $q_\pi(s, \pi'(s)) > v_\pi(s)$ for some s, then $v_{\pi'}(s) > v_\pi(s)$.

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- Proof of the theorem is not difficult for deterministic policies
- The theorem extends to probabilistic policies also
- Provides a basis to iteratively improve the policy

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$$\pi_0 \xrightarrow{\text{evaluate}} v_{\pi_0} \xrightarrow{\text{improve}} \pi_1 \xrightarrow{\text{evaluate}} v_{\pi_1} \xrightarrow{\text{improve}} \pi_2 \xrightarrow{\text{evaluate}} \cdots$$

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  - Must converge to optimal policy

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- Finite MDPs — can improve π only finitely many times,
  - Must converge to optimal policy
- Nested iteration — each policy evaluation is itself an iteration
  - Speed up by using v_{π_i} as initial state to compute $v_{\pi_{i+1}}$

Policy iteration

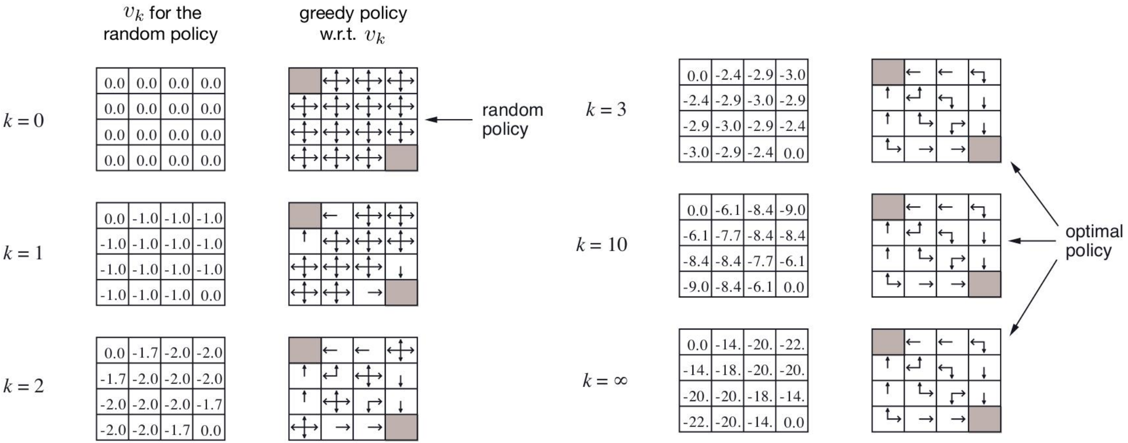
Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\quad \Delta \leftarrow 0$
   $\quad$ Loop for each $s \in \mathcal{S}$:
   $\qquad v \leftarrow V(s)$
   $\qquad V(s) \leftarrow \sum_{s',r} p(s', r \,|\, s, \pi(s))\big[r + \gamma V(s')\big]$
   $\qquad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $\quad$ *old-action* $\leftarrow \pi(s)$
   $\quad \pi(s) \leftarrow \operatorname{argmax}_a \sum_{s',r} p(s', r \,|\, s, a)\big[r + \gamma V(s')\big]$
   $\quad$ If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Optimizing Policy Iteration



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- Again, stop when incremental change $\Delta = |v_{\pi_{k+1}} - v_{\pi_k}|$ is below threshold θ

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- Asynchronous dynamic programming for large state spaces