Lecture 14: 5 March, 2024

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Data Mining and Machine Learning
January–April 2024

Mixture models

- Probabilistic process — parameters Θ
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  - Can we estimate p_1 and p_2?

Mixture models . . .

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- Sequence of N interleaved coin tosses $H\ T\ H\ H\ \cdots\ H\ H\ T$
- If the sequence is labelled, we can estimate p_1, p_2 separately
  - $H\ T\ T\ H\ H\ T\ H\ T\ H\ H\ T\ H\ T\ H\ T\ H\ H\ T\ H\ T$
  - $p_1 = 8/12 = 2/3$, $p_2 = 3/8$

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  - Make an initial guess for the parameters
  - Compute a (fractional) labelling of the outcomes
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  - $Pr(c_1 = T) = q_1 = 1/2$, $Pr(c_2 = T) = q_2 = 3/4$,

Assuming p_1, p_2

$P(H|c_1)$

$P(H|c_2)$

Want

$P(c_1|H)$

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  - For each H, likelihood it was c_i, $Pr(c_i \mid H)$, is $p_i/(p_1 + p_2)$

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  - For each T, likelihood it was c_i, $Pr(c_i \mid T)$, is $q_i/(q_1 + q_2)$
  - Assign fractional count $Pr(c_i \mid H)$ to each H: $2/3 \times c_1$, $1/3 \times c_2$

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  - For each T, likelihood it was c_i, $Pr(c_i \mid T)$, is $q_i/(q_1 + q_2)$
  - Assign fractional count $Pr(c_i \mid H)$ to each H: $2/3 \times c_1$, $1/3 \times c_2$
  - Likewise, assign fractional count $Pr(c_i \mid T)$ to each T: $2/5 \times c_1$, $3/5 \times c_2$

$$\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{4}} = \frac{2}{3}$$

$$\frac{\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}} = \frac{1}{3}$$

$$\frac{\frac{1}{2}}{\frac{1}{2}+\frac{3}{4}} = \frac{2}{5}$$

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- Initial guess: $p_1 = 1/2$, $p_2 = 1/4$
- Fractional counts: each H is $2/3 \times c_1$, $1/3 \times c_2$, each T: $2/5 \times c_1$, $3/5 \times c_2$

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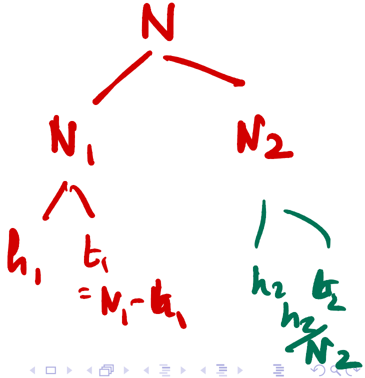
- *H T T H H T H T H H T H T H T H H T H T*

- Initial guess: $p_1 = 1/2$, $p_2 = 1/4$

- Fractional counts: each H is $2/3 \times c_1$, $1/3 \times c_2$, each T: $2/5 \times c_1$, $3/5 \times c_2$

- Add up the fractional counts
  - c_1: $11 \cdot (2/3) = \underline{22/3}$ heads, $9 \cdot (2/5) = \underline{18/5}$ tails $= N_1$
  - c_2: $11 \cdot (1/3) = \underline{11/3}$ heads, $9 \cdot (3/5) = 27/5$ tails $= N_2$



$$P_1 = \frac{h_1}{N_1}$$

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  - c_1: $11 \cdot (2/3) = 22/3$ heads, $9 \cdot (2/5) = 18/5$ tails
  - c_2: $11 \cdot (1/3) = 11/3$ heads, $9 \cdot (3/5) = 27/5$ tails
- Re-estimate the parameters
  - $p_1 = \dfrac{22/3}{22/3 + 18/5} = 110/164 = \underline{0.67}$, $q_1 = 1 - p_1 = \underline{0.33}$
  - $p_2 = \dfrac{11/3}{11/3 + 27/5} = 55/136 = \underline{0.40}$, $q_2 = 1 - p_2 = \underline{0.60}$

Repeat

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- Initial guess: $p_1 = 1/2$, $p_2 = 1/4$
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- Repeat until convergence

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- Observation $O = o_1 o_2 \ldots o_N$
- Expectation step
  - Compute likelihoods $Pr(M_i | o_j)$ for each M_i, o_j

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- Maximization step
  - Recompute MLE for each M_i using fraction of O assigned using likelihood

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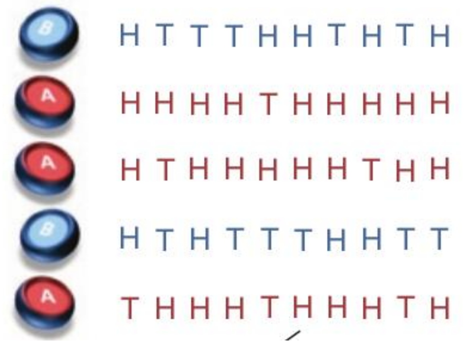
- Maximization step
  - Recompute MLE for each M_i using fraction of O assigned using likelihood
- Repeat until convergence
  - Why should it converge?
  - If the value converges, what have we computed?

EM — another example

- Two biased coins, choose a coin and toss 10 times, repeat 5 times



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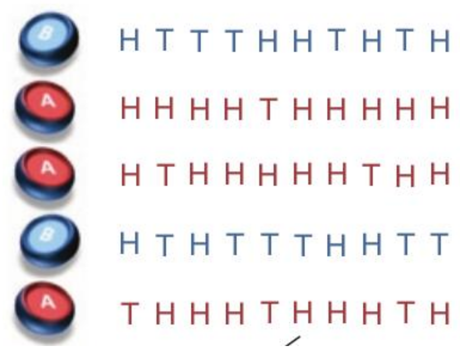


- If we know the breakup, we can separately compute MLE for each coin

Coin A	Coin B
	5 H, 5 T
9 H, 1 T	
8 H, 2 T	
	4 H, 6 T
7 H, 3 T	
24 H, 6 T	9 H, 11 T

$$\hat{\theta}_A = \frac{24}{24+6} = 0.80$$

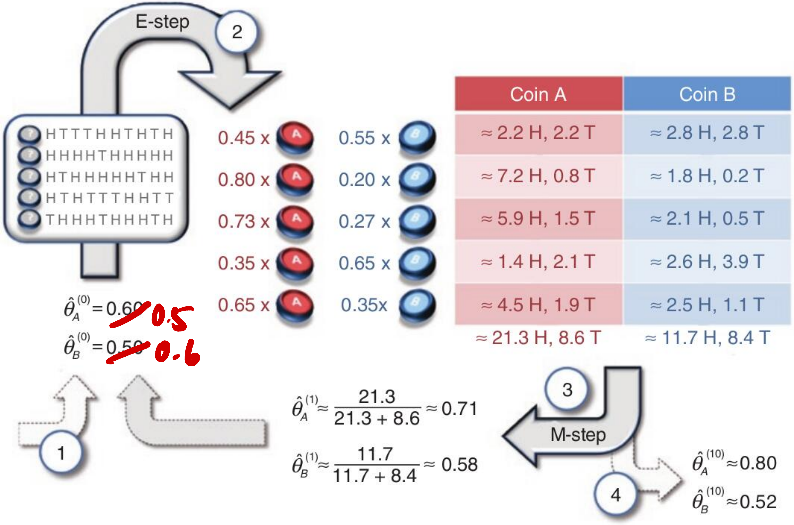
$$\hat{\theta}_B = \frac{9}{9+11} = 0.45$$

EM — another example

- Expectation-Maximization

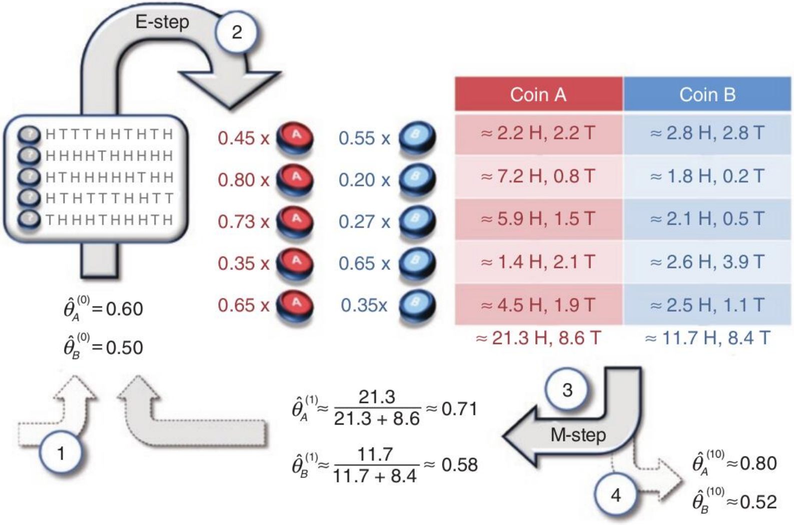
$$\begin{array}{cc} A & B \\ \frac{0.5}{1.1} & \frac{0.6}{1.1} \end{array}$$

$$= \quad 0.45 \quad 0.55$$



EM — another example

- Expectation-Maximization
- Initial estimates, $\theta_A = 0.6$, $\theta_B = 0.5$



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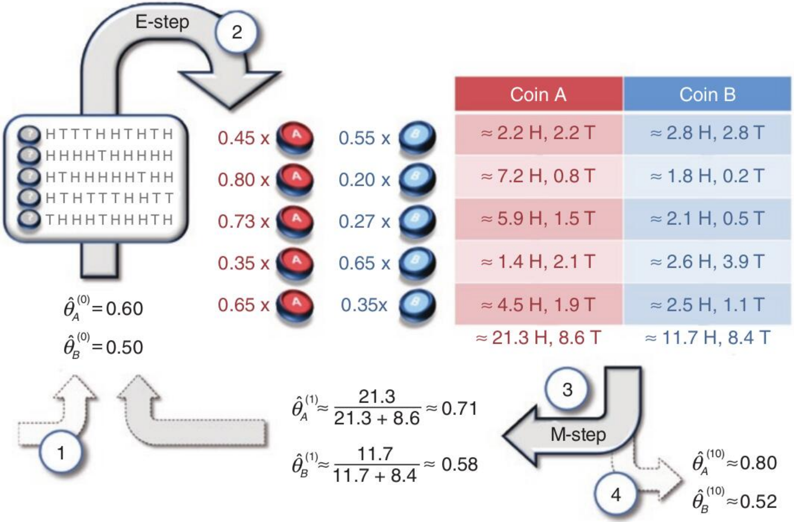
- Expectation-Maximization
- Initial estimates, $\theta_A = 0.6$, $\theta_B = 0.5$
- Compute likelihood of each sequence: $\theta^{n_H}(1-\theta)^{n_T}$



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- Expectation-Maximization
- Initial estimates, $\theta_A = 0.6$, $\theta_B = 0.5$
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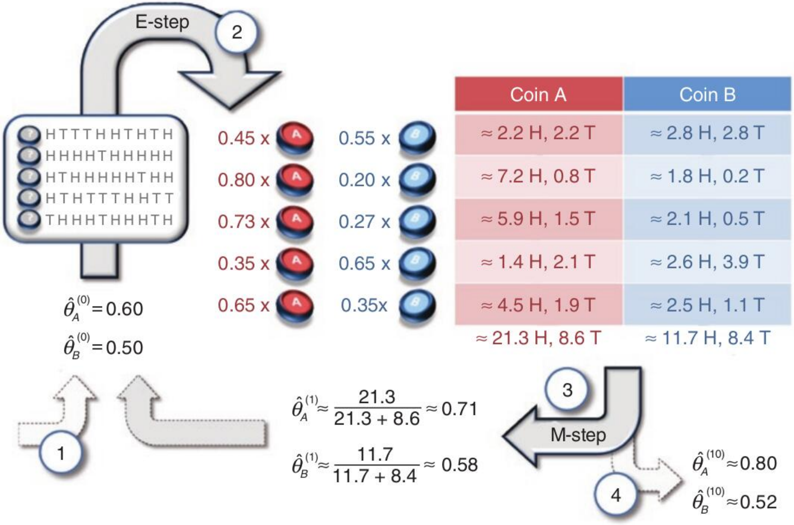
- Assign each sequence proportionately



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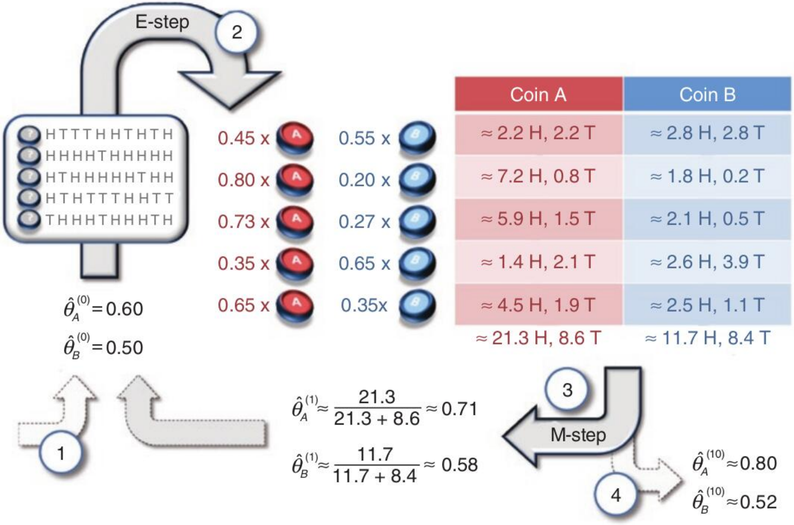
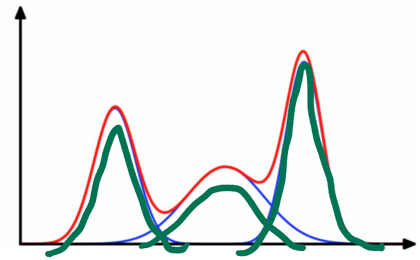
- Assign each sequence proportionately
- Converge to $\theta_A = 0.8$, $\theta_B = 0.52$



EM — mixture of Gaussians

- Sample uniformly from multiple Gaussians, $\mathcal{N}(\mu_i, \sigma_i)$



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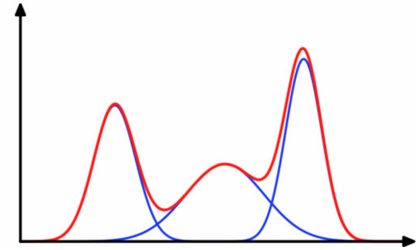

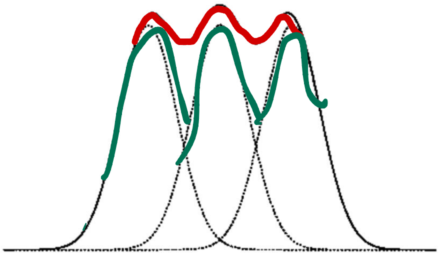

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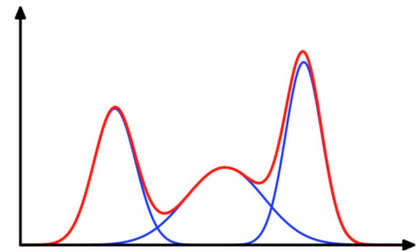

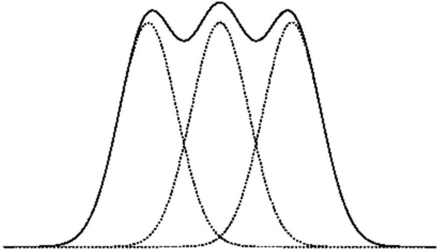

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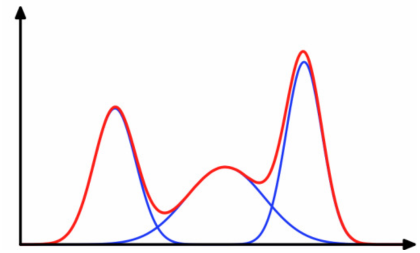

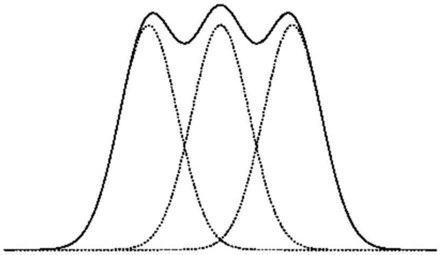

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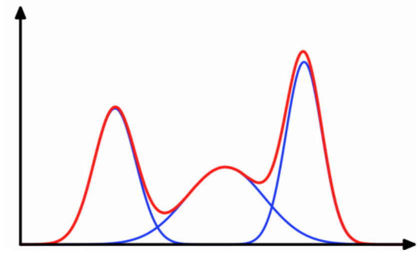

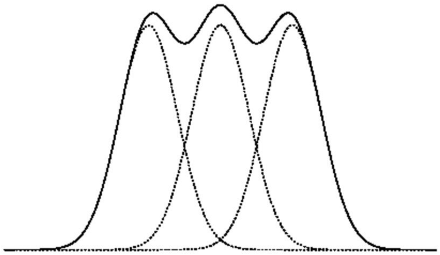

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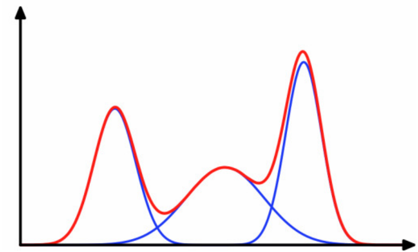
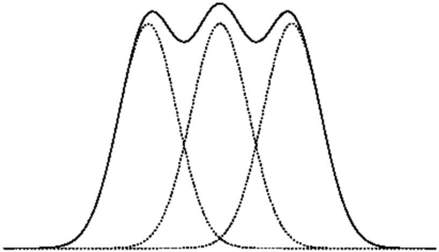
- $Pr(z_i \mid \mu_j) = exp(-\frac{1}{2\sigma^2}(z_i - \mu_j)^2)$
- $Pr(\mu_j \mid z_i) = c_{ij} = \dfrac{Pr(z_i \mid \mu_j)}{\sum_k Pr(z_i \mid \mu_k)}$





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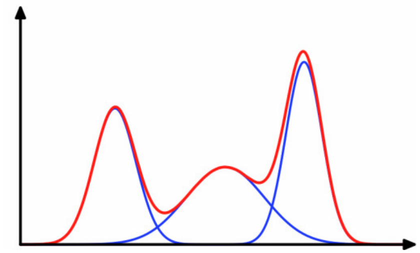
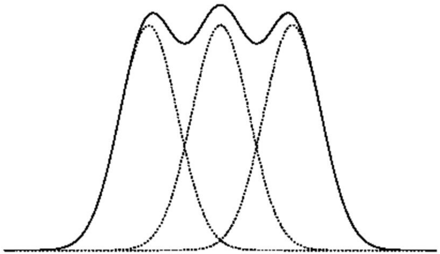
- $Pr(\mu_j \mid z_i) = c_{ij} = \dfrac{Pr(z_i \mid \mu_j)}{\sum_k Pr(z_i \mid \mu_k)}$
- MLE of μ_j is sample mean, $\dfrac{\sum_i c_{ij} z_i}{\sum_i c_{ij}}$





EM — mixture of Gaussians

- Sample uniformly from multiple Gaussians, $\mathcal{N}(\mu_i, \sigma_i)$
- For simplicity, assume all $\sigma_i = \sigma$
- N sample points $z_1, z_2, \ldots, z_N$
- Make an initial guess for each μ_j
- $Pr(z_i \mid \mu_j) = exp(-\frac{1}{2\sigma^2}(z_i - \mu_j)^2)$
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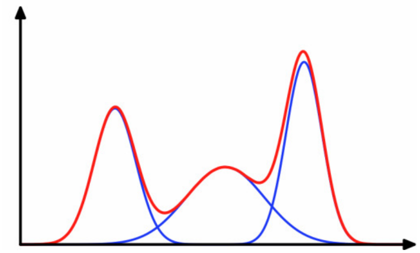
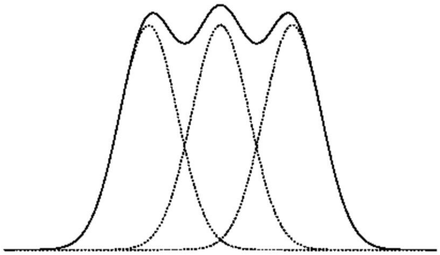
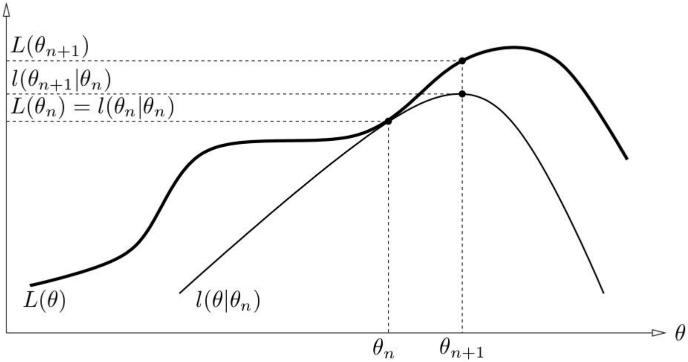
- Update estimates for μ_j and repeat





Theoretical foundations of EM

- Mixture of probabilistic models $(M_1, M_2, \ldots, M_k)$ with parameters $\Theta = (\theta_1, \theta_2, \ldots, \theta_k)$



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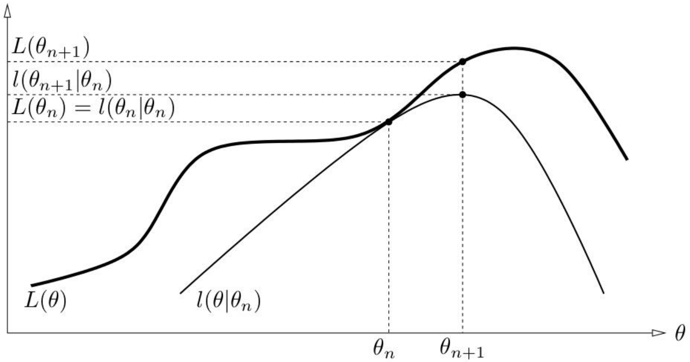

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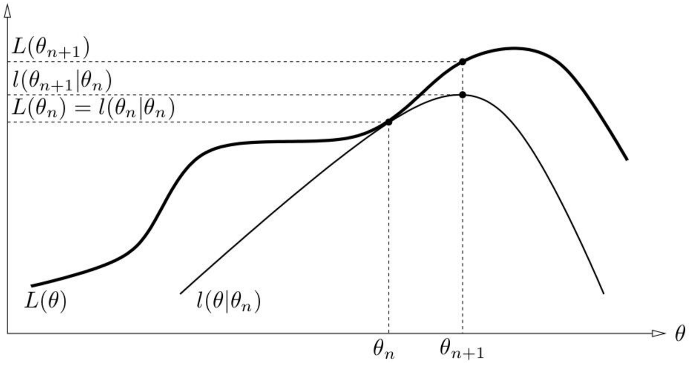

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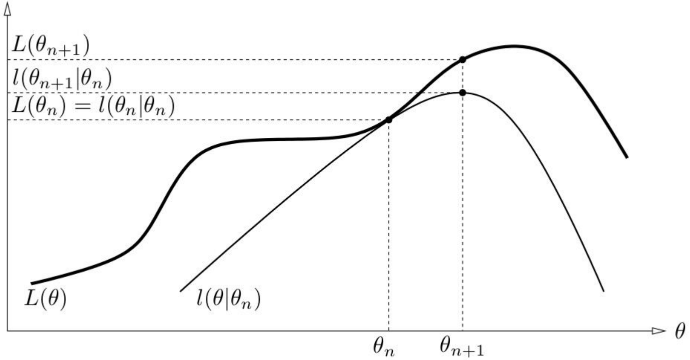

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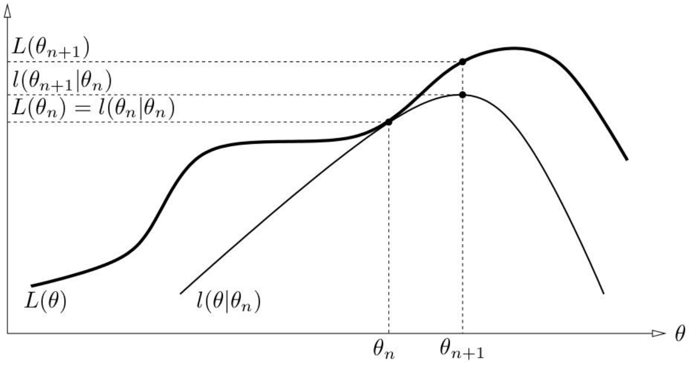

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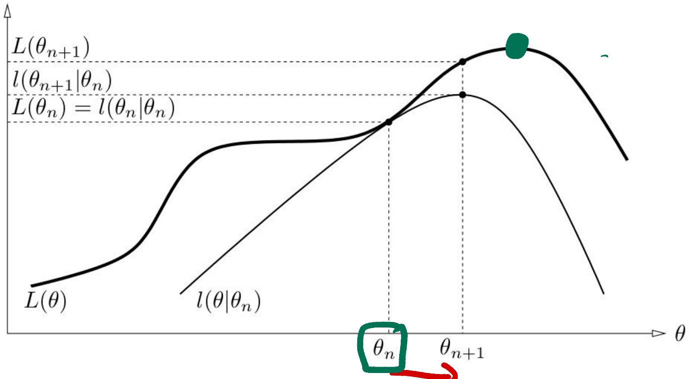

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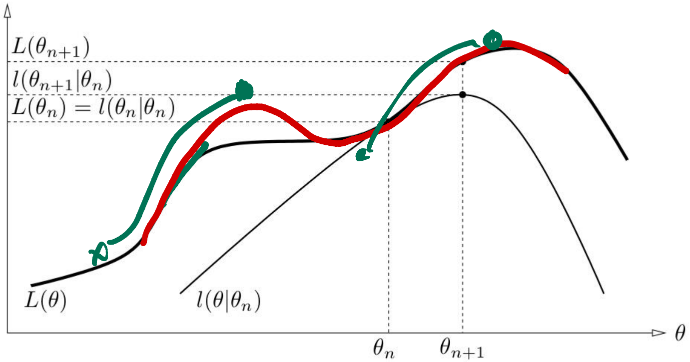

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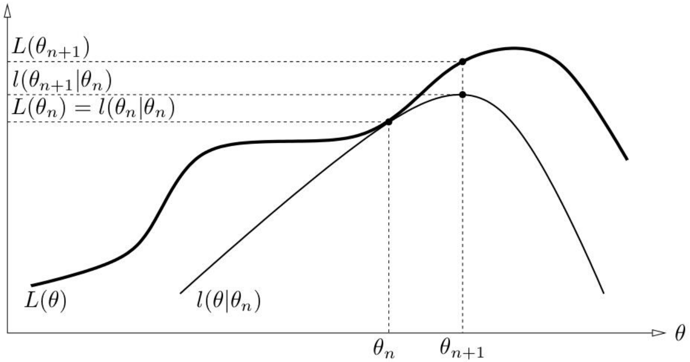

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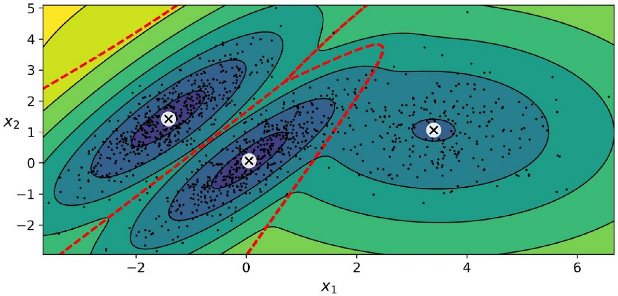
- Recompute $Pr(w_i \mid c_j)$ — fraction of occurrences of w_i in documents labelled c_j
  - n_{id} — occurrences of w_i in d
  - $Pr(w_i \mid c_j) = \dfrac{\sum_{d \in D} n_{id} \, Pr(c_j \mid d)}{\sum_{t=1}^{m} \sum_{d \in D} n_{td} \, Pr(c_j \mid d)}$

Clustering

- Data points from a mixture of Gaussian distributions



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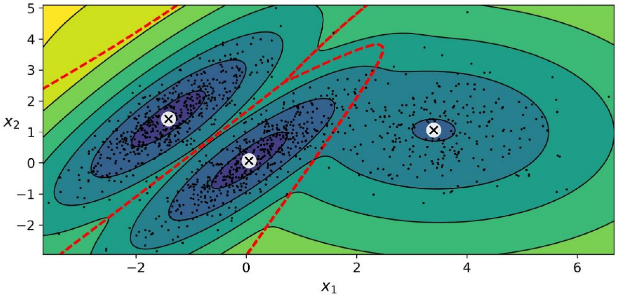

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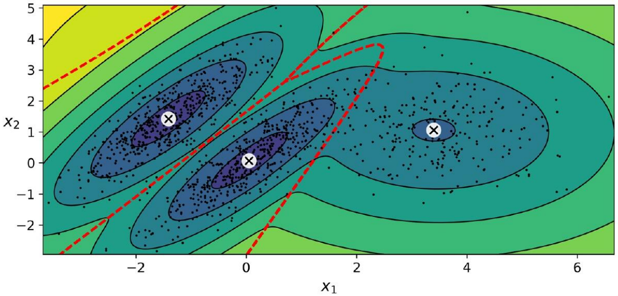

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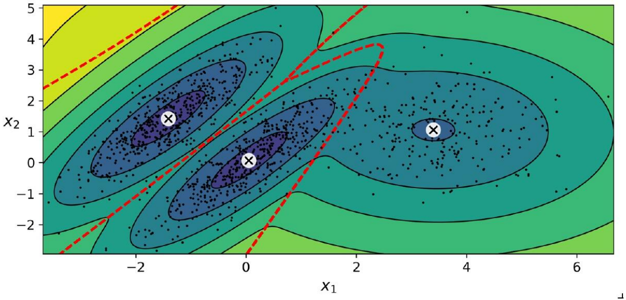

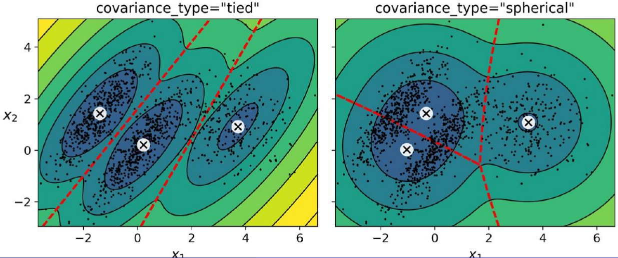

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