# Lecture 8: 1 February, 2024

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Data Mining and Machine Learning
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# Regression for classification

- Regression line
- Set a threshold
- Classifier
  - Output below threshold : 0 (No)
  - Output above threshold : 1 (Yes)

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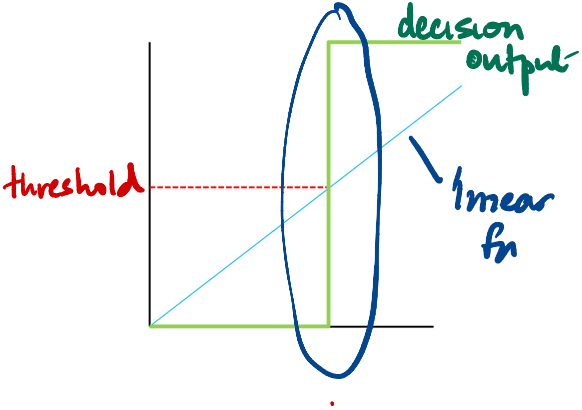
- Classifier output is a step function

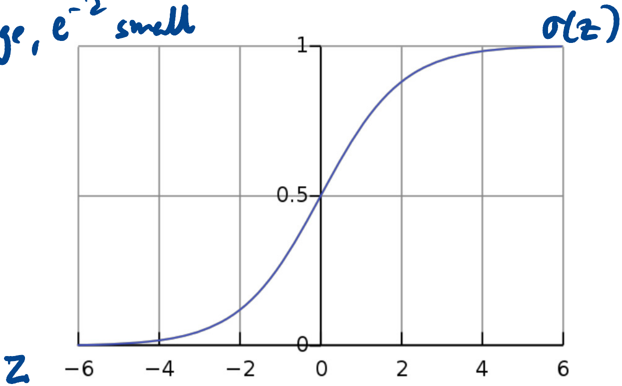


# Smoothen the step

- Sigmoid function

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

$z$ large, $e^{-z}$ small



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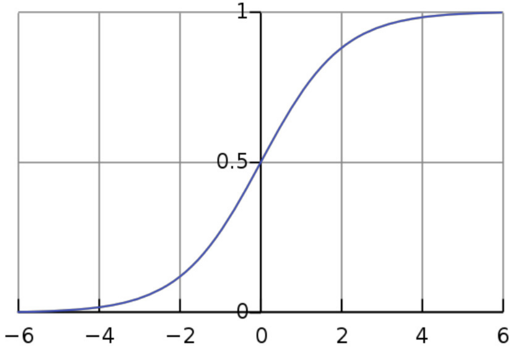
- Input $z$ is output of our regression

$$\sigma(z) = \frac{1}{1 + e^{-(\theta_0 + \theta_1 x_1 + \cdots + \theta_k x_k)}}$$



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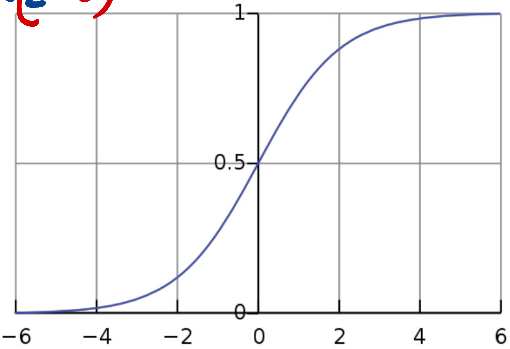
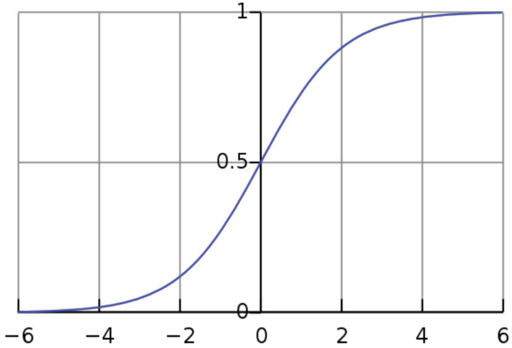
- Adjust parameters to fix horizontal position and steepness of step

$$\frac{1}{1+e^{-100(z-b)}}$$



# Logistic regression

- Compute the coefficients?
- Solve by gradient descent



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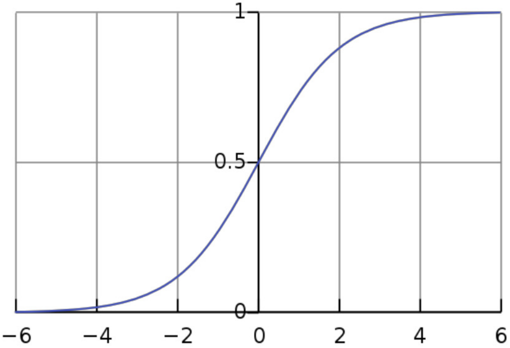
    - $\sigma'(z) = \sigma(z)(1 - \sigma(z))$



# Logistic regression

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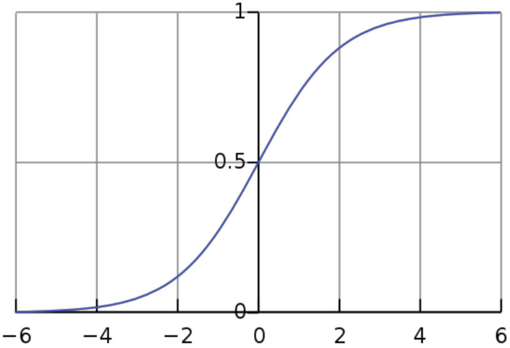
- Need a cost function to minimize



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- Combine as $P(y_i \mid x_i; \theta) = h_\theta(x_i)^{y_i} \cdot (1 - h_\theta(x_i))^{1-y_i}$

$y_i = 0 \rightarrow 1 - h_\theta(x_i)$

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  log + log

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$-\sum p_i \log p_i$

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- Log-likelihood: $\ell(\theta) = \sum_{i=1}^{n} y_i \log h_\theta(x_i) + (1 - y_i) \log(1 - h_\theta(x_i))$
- Minimize cross entropy: $-\sum_{i=1}^{n} y_i \log h_\theta(x_i) + (1 - y_i) \log(1 - h_\theta(x_i))$

# MSE for logistic regression and gradient descent

- Suppose we take mean sum-squared error as the loss function.
- Consider two inputs $x = (x_1, x_2)$

$$C = \frac{1}{n}\sum_{i=1}^{n}(y_i - \sigma(z_i))^2, \text{ where } z_i = \theta_0 + \theta_1 x_{i_1} + \theta_2 x_{i_2}$$

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  - For $j = 1, 2$,
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$$= \frac{2}{n}\sum_{i=1}^{n}(\sigma(z_i) - y_i)\sigma'(z_i)x_{i_j}$$

Chain rule

# MSE for logistic regression and gradient descent

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  - $\frac{\partial C}{\partial \theta_0} = \frac{2}{n}\sum_{i=1}^{n}(\sigma(z_i) - y_i)\frac{\partial \sigma(z_i)}{\partial z_i}\frac{\partial z_i}{\partial \theta_0} = \frac{2}{n}\sum_{i=1}^{n}(\sigma(z_i) - y_i)\sigma'(z_i) \cdot 1$

# MSE for logistic regression and gradient descent . . .

- For $j = 1, 2$, $\frac{\partial C}{\partial \theta_j} = \frac{2}{n}\sum_{i=1}^{n}(\sigma(z_i) - y_i)\sigma'(z_i)x_j^i$, and $\frac{\partial C}{\partial \theta_0} = \frac{2}{n}\sum_{i=1}^{n}(\sigma(z_i) - y_i)\sigma'(z_i)$
- Each term in $\frac{\partial C}{\partial \theta_1}, \frac{\partial C}{\partial \theta_2}, \frac{\partial C}{\partial \theta_0}$ is proportional to $\sigma'(z_i)$

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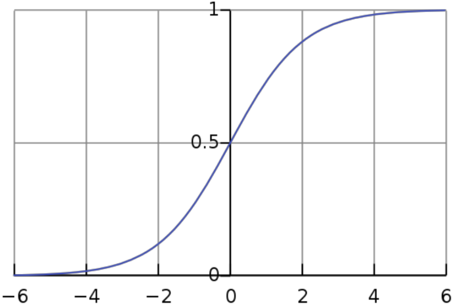
- Each term in $\dfrac{\partial C}{\partial \theta_1}, \dfrac{\partial C}{\partial \theta_2}, \dfrac{\partial C}{\partial \theta_0}$ is proportional to $\sigma'(z_i)$
- Ideally, gradient descent should take large steps when $\sigma(z) - y$ is large
- $\sigma(z)$ is flat at both extremes
- If $\sigma(z)$ is completely wrong, $\sigma(z) \approx (1 - y)$, we still have $\sigma'(z) \approx 0$
- Learning is slow even when current model is far from optimal



# Cross entropy and gradient descent

- $C = -[y \ln(\sigma(z)) + (1-y)\ln(1-\sigma(z))]$

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- $\frac{\partial C}{\partial \theta_j} = \frac{\partial C}{\partial \sigma}\frac{\partial \sigma}{\partial \theta_j} = -\left[\frac{y}{\sigma(z)} - \frac{1-y}{1-\sigma(z)}\right]\frac{\partial \sigma}{\partial \theta_j}$

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$$= -\left[\frac{y}{\sigma(z)} - \frac{1-y}{1-\sigma(z)}\right]\sigma'(z)x_j$$

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- $\frac{\partial C}{\partial \theta_j} = -\left[\frac{y(1-\sigma(z)) - (1-y)\sigma(z)}{\sigma(z)(1-\sigma(z))}\right]\sigma'(z)x_j$
- Recall that $\sigma'(z) = \sigma(z)(1-\sigma(z))$

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- Similarly, $\dfrac{\partial C}{\partial \theta_0} = (\sigma(z) - y)$
- Thus, as we wanted, the gradient is proportional to $\sigma(z) - y$

Logistic regression

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- Similarly, $\frac{\partial C}{\partial \theta_0} = (\sigma(z) - y)$
- Thus, as we wanted, the gradient is proportional to $\sigma(z) - y$
- The greater the error, the faster the learning rate