Smooth torus quotients of Schubert varieties in the Grassmannian

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Abstract

Let $r < n$ be positive integers and further suppose r and n are coprime. We study the GIT quotient of Schubert varieties $X(w)$ in the Grassmannian $G_{r,n}$, admitting semistable points for the action of T with respect to the T-linearized line bundle $\mathcal{L}$. We give necessary and sufficient combinatorial conditions for the GIT quotient $T \backslash\backslash X(w)^{ss}_T(\mathcal{L})$ to be smooth.

1 Introduction

Let $G_{r,n}$ denote the Grassmannian variety of r dimensional subspaces of $\mathbb{C}^n$. Let $G = SL(n, \mathbb{C})$. Let T be the maximal torus of G consisting of all diagonal matrices in G. Let $\phi : G_{r,n} \longrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n)$ denote the Plücker embedding. Let $\mathcal{L}$ be the pullback line bundle $\phi^*(\mathcal{O}(1)^{\otimes n})$. Let

$$I(r, n) := \{(i_1, i_2, ..i_r)|1 \leq i_1 < i_2 \ldots . < i_r \leq n\}.$$

For $w \in I(r, n)$, let $X(w) \subset G_{r,n}$ denote the Schubert variety corresponding to w (see [8]).

The line bundle $\mathcal{L}$ is T-linearised. Let $X(w)^{ss}_T(\mathcal{L})$(respectively, $X(w)^{ss}_T(\mathcal{L})$) be the set of semistable (respectively, stable) points with respect to the T-linearised line bundle $\mathcal{L}$(see [10]). It is known from the work of Kumar [6] and also the work of Kannan-Sardar [5] that the line bundle $\mathcal{L}$ descends to the GIT quotient $T \backslash\backslash X(w)^{ss}_T(\mathcal{L})$ precisely when $n|d$.

Kannan and Sardar [5] showed that there is a unique minimal Schubert Variety $X(w_{r,n})$ in $G_{r,n}$ admitting semistable points with respect to the line bundle $\mathcal{L}$. We denote by $(a_1, a_2, \ldots, a_r)$ the unique representative of $w_{r,n}$ in $I(r, n)$ (see also Proposition 2.1).

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Our paper is motivated by the question of when the GIT quotient $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth. An understanding of the GIT quotient in the case $\gcd(r,n) \neq 1$ is difficult since stability is different from semistability. So we assume that $\gcd(r,n) = 1$. Under this assumption Skorobogotov (see [11]) and Kannan(see [4]) showed that the quotient variety $T\backslash\backslash (G_{r,n})^{ss}_T(\mathcal{L})$ is smooth. In [1], it was shown that the GIT quotient $T\backslash\backslash X(w_{r,n})^{ss}_T(\mathcal{L})$ is smooth. In this paper we prove the following theorem:

Theorem 1.1. *Let $w = (b_1, b_2, \ldots, b_r) \in I(r,n)$ with $b_i \geq a_i$ for all $1 \leq i \leq r$. Let $X(v_1), \ldots, X(v_k)$, be the k components in the singular locus of $X(w)$. Then the following are equivalent*

(1) *$T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth.*

(2) *For all i, we have $v_i \not\geq w_{r,n}$.*

(3) *Whenever $b_j \geq b_{j-1} + 2$ we have $a_j \geq b_{j-1} + 1$.*

In other words the GIT quotient is smooth precisely when the semistable locus does not intersect the singular locus, and there is a simple combinatorial criterion describing when this happens.

2 Notations and Preliminaries

In this section we set up some notation which will be needed to prove the theorem. We also recall some results on the singular locus of Schubert varieties in the Grassmannian.

Let G and T be as above. Let B be the Borel subgroup of all upper triangular matrices in G. Let R denotes the root system of G with respect to T. Let $R^+ \subset R$ be the set of positive roots with respect to B. Let $S = \{\alpha_1, \ldots \alpha_{n-1}\} \subset R^+$ be the set of simple roots. Let $N_G(T)$ denote the normaliser of T in G. Let $W = N_G(T)/T$ denote the Weyl group. Let s_i be the simple reflection corresponding to α_i. Let $n_i \in N_G(T)$ be a representative of s_i. The unipotent group associated to a root $\beta \in R$ is denoted by U_β. For $J \subset S$ let P_J be the subgroup of G generated by B and $\{n_i : \alpha_i \in J\}$. Let $W^J := \{w \in W : w(\alpha) > 0$ for all $\alpha \in J\}$. $P_{\hat{\alpha_r}}$ denotes the maximal parabolic subgroup corresponding to $J = S \setminus \{\alpha_r\}$. We identify $W^{S\setminus\{\alpha_r\}}$ with $I(r,n)$ where $w \mapsto (w(1), \ldots, w(r))$. The following proposition was first proved by Kannan and Sardar, [5]. A simpler proof was given in [1].

Proposition 2.1. *[1] Let r and n be coprime. There is a unique minimal Schubert Variety $X(w_{r,n})$ in $G_{r,n}$ admitting semistable points with respect to the T-linearized line bundle $\mathcal{L}$. As an element of $I(r,n)$, $w_{r,n} = (a_1, a_2, \ldots, a_r)$ where a_i is the smallest integer such that $a_i \cdot r \geq i \cdot n$.*

Let $w = (b_1, \ldots, b_r)$. Clearly $X(w)$ has semistable points with respect to the T-linearized line bundle $\mathcal{L}$ if and only if $b_i \geq a_i$ for all i.

Now if $w = (b_1, \dots, b_r) \in I(r,n)$, one reduced expression for the Weyl group element in $W^{S \setminus \{\alpha_r\}}$ corresponding to w is $(s_{b_1-1} \cdots s_1) \dots (s_{b_i-1} \cdots s_i) \dots (s_{b_r-1} \cdots s_r)$ where a bracket is assumed to be empty is if $b_i - 1$ is less than i. Since $a_i \geq i+1$ for all $1 \leq i \leq r$, we have

$$w_{r,n} = (s_{a_1-1} \cdots s_1)(s_{a_2-1} \cdots s_2) \dots (s_{a_r-1} \cdots s_r),$$

and no bracket is empty in the above expression.

2.1 Smooth locus of Schubert varieties in $G_{r,n}$

The singular loci of Schubert varieties in miniscule G/P were determined by Lakshmibai and Weyman [9]. There is another description of the singular locus of Schubert varieties $X(w)$ in terms of the stabiliser parabolic subgroup of $X(w)$, due to Brion and Polo [2]. They proved the following theorem.

Theorem 2.2. *Let $w \in I(r,n)$. Let $P_w = \{g \in G | gX(w) = X(w)\}$, the stabilizer of $X(w)$ in G. The smooth locus of the Schubert variety $X(w)$ is $X(w)_{sm} = P_w wP/P \subseteq X(w) \subseteq G_{r,n}$.*

We recall the following proposition from [7].

Proposition 2.3. *Let $w = (b_1, b_2, \dots, b_r) \in I(r,n)$. Define*

$$J'(w) := \{j \in [1, \dots, n-1] \big| \exists m \text{ with } j = b_m, j+1 \neq b_{m+1}\}.$$

Let $J(w) := \{1, 2, \dots, n-1\} \setminus J'(w)$. Then $P_w = P_J$ where $J = \{\alpha_j | j \in J(w)\}$

We need some more notation to describe the work in [9]. Let $w = (b_1, b_2, \dots, b_r)$. Associate to w the increasing sequence $\mathbf{w} = (\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_r})$ where $\mathbf{b_i} = b_i - i$, so $0 \leq \mathbf{b_1} \leq \mathbf{b_2} \leq \dots \leq \mathbf{b_r} \leq n-r$. Clearly we have a bijective correspondence between $I(r,n)$ and non-decreasing r length sequences in $0 \leq \mathbf{b_1} \leq \mathbf{b_2} \leq \dots \leq \mathbf{b_r} \leq n-r$. An increasing sequence gives us a Young diagram, $Y(\mathbf{w})$, in an $r \times n-r$ rectangle with the i-th row having $\mathbf{b_i}$ boxes[1]. We call this the Young diagram corresponding to the Schubert variety $X(w)$.

Recall the following Theorem from [9][2].

Theorem 2.4 (Theorem 5.3 [9])**.** *Let $X(w)$ be a Schubert variety in the Grassmannian. Let $\mathbf{w} = (\mathbf{p_1^{q_1}}, \dots, \mathbf{p_k^{q_k}}) = (\underbrace{p_1, \dots p_1}_{q_1 \text{ times}} \dots, \underbrace{p_k, \dots p_k}_{q_k \text{ times}})$ be the non-zero parts of the increasing sequence $\mathbf{w}$ with $1 \leq \mathbf{p_1} < \mathbf{p_2} \dots < \mathbf{p_k} \leq n-r$. The singular locus $X(w)$ consists of $k-1$ components. The components are given by the Schubert varieties corresponding to the Young diagrams $Y(\mathbf{w_1}), \dots Y(\mathbf{w_{k-1}})$, where the sequences $\mathbf{w_i}$ are given by*

$$\mathbf{w_i} = (p_1^{q_1}, \dots, p_{i-1}^{q_{i-1}}, (p_i - 1)^{q_i+1}, p_{i+1}^{q_{i+1}-1}, p_{i+2}^{q_{i+2}}, \dots, p_r^{q_r}),$$

for $1 \leq i \leq r-1$ and $1 \leq p_i < p_{i+1}$.

[1] rows are numbered $1, \dots, r$ from bottom to top.

[2] the notation we use is different from theirs, they work with non-increasing sequences

An inner corner in a Young diagram is a box that, if it is removed, still gives us the Young diagram of an non-decreasing sequence. So an easy to remember description of the irreducible components of the singular locus of $X(w)$ is as follows :- they are the Schubert varieties in correspondence with Young diagram $Y(\mathbf{w_i})$ obtained from $Y(\mathbf{w})$ by removing the hook from the i-th inner box to the $i+1$-st inner box.

3 Smoothness of GIT quotients

In this section we first give a criterion for the GIT quotient to be smooth. We then prove the main theorem by showing that if the combinatorial conditions in the statement of the main theorem hold this criterion is met. We assume that r and n coprime.

Proposition 3.1. *Let $w \in I(r,n)$. $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth if and only if $X(w)^{ss} \subseteq X(w)_{sm}$.*

Proof. Assume that $X(w)^{ss} \subseteq X(w)_{sm}$. Since $gcd(r,n) = 1$, it follows from [11, Corollary 2.5] and [3, Theorem 3.3] that $X(w)^{ss} = X(w)^{s}$. So the stablizer of all semistable points $x \in X(w)^{ss}$ is finite. The proof now follows along the lines described in [4]. Suppose $x \in BvP/P$ for some v. Let $R^+(v^{-1})$ denote the set of all positive roots made negative by v^{-1}. And choose a subset $\beta_1, \ldots, \beta_k$ of positive roots in $R^+(v^{-1})$ such that $x = u_{\beta_1}(t_1)\ldots, u_{\beta_k}(t_k)vP/P$ with $u_{\beta_j}(t_j)$ in the root subgroup U_{β_j}, $t_j \neq 0$ for $j = 1, \ldots, k$. The isotropy group T_x is $\cap_{i=1}^{i=k}\ker(\beta_j)$. Since this is finite, it follows from [4, Example 3.3] that $T_x = Z(G)$, the center of G. Working with the adjoint group we may assume that the stablizer is trivial. So $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth.

For the converse, first note that since we are in the case $\gcd(r,n) = 1$ the quotient is a geometric quotient i.e. it is an orbit space. But then restricted to $X(w)^{ss}$ the quotient is a T-bundle. So smooth points go to smooth points in the quotient and singular points go to singular points. Since the quotient is smooth it follows that each point $x \in X(w)^{ss}$ is smooth in $X(w)^{ss}$. Since $\mathcal{O}_{x,X(w)^{ss}} = \mathcal{O}_{x,X(w)}$, it follows that $X(w)^{ss} \subseteq X(w)_{sm}$. □

We prove the main theorem.

Theorem 3.2. *Let $w = (b_1, b_2, \ldots, b_r) \in I(r,n)$ with $b_i \geq a_i$ for all $1 \leq i \leq r$. Let $X(v_1), \ldots, X(v_k)$, be the k components in the singular locus of $X(w)$. Then the following are equivalent*

(1) *$T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth.*

(2) *For all i, we have $v_i \not\geq w_{r,n}$.*

(3) *Whenever $b_j \geq b_{j-1}+2$ we have $a_j \geq b_{j-1}+1$.*

Proof. Since $b_i \geq a_i$ for all i we have that $X(w)^{ss}_T(\mathcal{L})$ is non empty. From Proposition 3.1, $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth if and only if $w_i \not\geq w_{r,n}$ for all i. Hence the equivalence of (1) and (2).

We prove the equivalence of (2), (3). The components of the singular locus of $X(w)$ are Schubert varieties $X(w_i)$ in correspondence with diagrams obtained from $Y(\mathbf{w})$ by removing hooks. There is a hook at row j of $Y(\mathbf{w})$ if and only if $b_j \geq b_{j-1}+2$. We denote the Schubert variety obtained from w by removing the hook at row j by $X(w_j)$. Let the word corresponding to it in $I(r,n)$ be $(b'_1, b'_2, \ldots, b'_r)$. Now $X(w)$ contains $X(w_{r,n})$. Let t be the smallest integer less than j such that $b_{k+1} = b_k + 1$ for all $t \leq k < j$. By definition of w_j we have

$$b'_p = \begin{cases} b_p & 1 \leq p \leq t-1, \\ b_p - 1 & t \leq p \leq j-1, \\ b_{j-1} & p = j, \\ b_p & j+1 \leq p \leq r. \end{cases}$$

Now $X(w)$ contains $X(w_{r,n})$ so $b_p \geq a_p$ for $1 \leq p \leq r$. So $X(w_j)$ does not contain $X(w_{r,n})$ if and only if $a_p \geq b'_p + 1 = b_p$, for some $t \leq p \leq j-1$ or if $a_j \geq b'_j + 1 = b_{j-1}+1$. Now $b_p = b_{p-1}+1$ for all $t < p \leq j-1$, and $a_p \geq a_{p-1}+1$. It follows that if for some $t \leq p \leq j-1$, $a_p \geq b_p$ then $a_{p+1} \geq a_p + 1 \geq b_p + 1 = b_{p+1}$, and so we conclude that $a_{j-1} \geq b_{j-1}$. Then $a_j \geq a_{j-1} + 1 \geq b_{j-1} + 1$, completing the proof.

An alternate proof of the equivalence of (2), (3) is as follows. First assume that for all j for which $b_j \geq b_{j-1}+2$ we have $a_j \geq b_{j-1}+1$. Now let $u \in W^P$ be such that $w_{r,n} \leq u \leq w$. Let the one line notation for u be $(b'_1, b'_2, \ldots, b'_r)$. Then $a_i \leq b'_i \leq b_i$ for all $1 \leq i \leq r$. Define $u_i = s_{b'_i} s_{b'_i+1} \ldots s_{b_i-1}$ for $1 \leq i \leq r$. Clearly $u = u_1(s_{b_1-1}\cdots s_1)u_2(s_{b_2-1}\cdots s_2)\cdots u_r(s_{b_r-1}\cdots s_r)$.

For every $1 < i \leq r$, the index of the least simple reflection less than u_i in the Bruhat order is $s_{b'_i}$ and the index of the largest simple reflection less than u_{i-1} in the Bruhat order is $s_{b_{i-1}-1}$. Take any $1 \leq i \leq r$ for which $b_i \geq b_{i-1}+2$. By our hypothesis we have $b'_i \geq b_{i-1}+1$, so $s_{b_{i-1}} \not\leq u_i$ and $u_i \in P_w$ from Proposition 2.3. Further u_i and u_{i-1} commute. For each $1 < i \leq r$ for which $b_i = b_{i-1}+1$, $u_i \in P_w$ from Proposition 2.3. Clearly $u_1 \in P_w$. So for all i, $u_i \in P_w$. It is easy to check that $u = u_r u_{r-1} \ldots u_2 u_1 w P/P$. Therefore, by Theorem 2.2, $X(v) \subset X(w)_{sm}$.

Now assume that $b_j \geq b_{j-1}+2$ but $a_j \leq b_{j-1}$. iI follows from the definition of $J'(w)$ in Proposition 2.3 that $b_{j-1} \in J'(w)$. Let t be the smallest integer less than j such that $b_{k+1} = b_k + 1$ for all $t \leq k < j$. Then w has a reduced expression of the form

$$w = w" s_{b_t-1} \ldots s_t s_{b_{t+1}-1} \ldots s_{t+1} \ldots s_{b_{j-1}-1} \ldots s_{j-1} s_{b_j-1} \ldots s_j w'.$$

Now consider the Weyl group element

$$u = w" s_{b_t-2} \ldots s_t s_{b_{t+1}-2} \ldots s_{t+1} \ldots s_{b_{j-1}-2} \ldots s_{j-1} s_{b_{j-1}-1} \ldots s_j w'.$$

Clearly $u \leq w$ and u is obtained from w by left multiplying with the reduced word

$$s_{b_{j-1}} s_{b_{j-1}+1} \ldots s_{b_j-2} s_{b_j-1} \ldots s_{b_{t+2}-1} s_{b_{t+1}-1} s_{b_t-1}.$$

In the one line notation $u = (b_1, \ldots, b_{t-1}, b_t - 1, \ldots, b_{j-1} - 1, b_{j-1}, b_{j+1}, \ldots, b_r)$. Note that $J'(u) \subseteq J'(w)$, so $P_w \subseteq P_u$ and therefore P_w stabilises $X(u)$. Since $u < w$, w is not element of $X(u)$. And so $wP/P \notin P_w uP/P$. Hence $uP/P \notin P_w wP/P$. Therefore, by Theorem 2.2, $X(u)$ is in the singular locus of $X(w)$. However, if $a_j \leq b_{j-1}$, it can be easily seen that $u \geq w_{r,n}$, implying that $X(u)$ contains a semistable point, a contradiction. □

4 Examples and non-examples

We illustrate the proof of the main theorem with a simple example.

Example 4.1. *Consider the Schubert variety corresponding to $w = (3, 5, 7, 9)$ in $I(4, 9)$. The Young diagram associated to w is given by the increasing sequence $\mathbf{w} = (\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5})$. Fill this diagram starting with s_i at the leftmost box in row i, and filling the boxes to the right of this entry in row i with $s_{i+1}, s_{i+2}, \ldots$, in order, all the way to the last box in row i. We get the filling*

s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	
s_2	s_3	s_4		
s_1	s_2			

Reading the entries in the Young diagram from right to left in each row, and bottom to top yields $s_2s_1s_4s_3s_2s_6s_5s_4s_3s_8s_7s_6s_5s_4$, the element in W^P corresponding to the Schubert variety $(3, 5, 7, 9)$. According to Theorem 2.4 the singular locus of this Schubert variety has three irreducible components given by the sequences $(1, 1, 4, 5)$,$(2, 2, 2, 5)$ and $(2, 3, 3, 3)$. The corresponding Schubert varieties are given by the tuples $(2, 3, 7, 9)$, $(3, 4, 5, 9)$ and $(3, 5, 6, 7)$, respectively. The Weyl group elements corresponding to these varieties are $s_1s_2s_6s_5s_4s_3s_8s_7s_6s_5s_4$, $s_2s_1s_3s_2s_4s_3s_8s_7s_6s_5s_4$ and $s_2s_1s_4s_3s_2s_5s_4s_3s_6s_5s_4$ respectively. Note that these words can be obtained by removing the hooks occupied by $s_2s_3s_4$, $s_4s_5s_6$ and $s_6s_7s_8$, respectively, and reading the entries left in the resulting Young diagrams from bottom to top, and right to left in each row - exactly as we did for w.

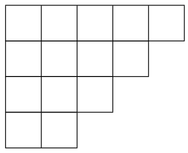
Let us show for example that the Schubert variety corresponding to the Weyl group element $v = s_1s_2s_6s_5s_4s_3s_8s_7s_6s_5s_4$ is not in the smooth locus by showing that it does not satisfy the hypothesis of Theorem 2.2. The stabilizer of $X(w)$ is the parabolic subgroup corresponding to the subset of simple reflections $\{\alpha | s_\alpha w \leq w\}$. In this case it can be checked that this is the parabolic subgroup corresponding to $\{\alpha_1, \alpha_2, \alpha_4, \alpha_6, \alpha_8\}$ which is $P_{\hat{\alpha_3}} \cap P_{\hat{\alpha_5}} \cap P_{\hat{\alpha_7}}$. However v is obtained from w by multiplying on the left with $s_3s_4s_2$. And this element is not in $P_{\hat{\alpha_3}} \cap P_{\hat{\alpha_5}} \cap P_{\hat{\alpha_7}}$. It can be similarly shown that the other two components are also not in the smooth locus - the Weyl group elements corresponding to them are obtained from w by multiplying on the left with $s_4s_6s_5$ and $s_6s_8s_7$ respectively and these elements are clearly not in $P_{\hat{\alpha_3}} \cap P_{\hat{\alpha_5}} \cap P_{\hat{\alpha_7}}$.

We conclude with examples of Schubert varieties in $G_{4,9}$ whose GIT quotients are singular, and examples of Schubert varieties whose GIT quotients are smooth.

Example 4.2. *We know from 2.1 that* $w_{4,9} = (3, 5, 7, 9)$*. A reduced expression for the word* $w_{4,9}$ *is*

$$s_2 s_1 s_4 s_3 s_2 s_6 s_5 s_4 s_3 s_8 s_7 s_6 s_5 s_4.$$

The Young diagram $Y(\mathbf{w_{4,9}})$ *corresponding to* $w_{4,9}$ *is*



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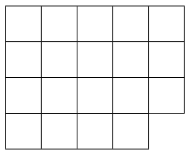
Recall from Theorem 3.1 [1] we have $T \backslash\backslash X(w)^{ss}_T(\mathcal{L})$ *is smooth.*

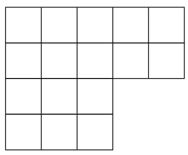
Example 4.3. *Let us consider the word* $w = (5, 7, 8, 9)$*. A reduced expression for* w *is*

$$s_4 s_3 s_2 s_1 s_6 s_5 s_4 s_3 s_2 s_7 s_6 s_5 s_4 s_3 s_8 s_7 s_6 s_5 s_4.$$

The Young diagram $Y(\mathbf{w})$ *is*



The singular locus $X(w)$*, obtained by removing the only hook corresponds the following tableau:*

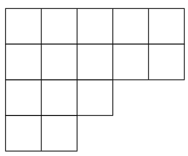


Here $w' = (4, 5, 8, 9)$*. Since* $w' > w_{4,9}$*,* $X(w')$ *contains semistable points and hence the quotient space* $T \backslash\backslash X(w)^{ss}_T(\mathcal{L})$ *is not smooth (using 3.2).*

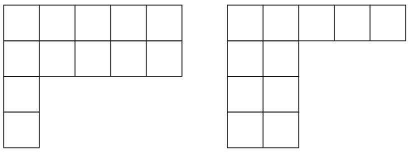
Example 4.4. *Consider the word* $w = (3, 5, 8, 9)$*. A reduced expression for* w *is*

$$s_2 s_1 s_4 s_3 s_2 s_7 s_6 s_5 s_4 s_3 s_8 s_7 s_6 s_5 s_4.$$

The Young diagram $Y(\mathbf{w})$ *is*



The singular locus obtained by removing the hooks has Schubert varieties $X(w_1), X(w_2)$, whose Young diagrams are given by the following tableaux.



Here $w_1 = (2, 3, 8, 9)$ and $w_2 = (3, 4, 5, 9)$. Note for $i = 1, 2$ $w_i \not\geq w_{4,9}$, so neither $X(w_1)$ nor $X(w_2)$ contain semistable points. Hence the quotient space $T\backslash\!\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth (using Theorem 3.2).

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