

On the internal tensor product of polynomial functors.

Upendra Kulkarni* Shraddha Srivastava† K. V. Subrahmanyam ‡

March 31, 2015

Abstract

Polynomial representations of the general linear group $GL(V)$ have been studied for a over a century going back to Schur. It has been understood for a long time that these are functorial in V . Subsequently strict polynomial functors were defined and used for several applications.

More recently, Krause [Kra13] has defined, via Day convolution, a very interesting tensor product on the category of strict polynomial functors of a fixed degree over a commutative ring. It is well known that there is an exact functor, the *Schur functor*, from the category of polynomial representations/functors of fixed degree d to that of representations of S_d , the symmetric group on d letters.

We show that the Schur functor takes internal tensor product to the usual Kronecker tensor product of representations of S_d , i.e. where the group elements act diagonally. Our proof relies on examining the effect of the Schur functor on the projective generator given by parametrized divided powers $\Gamma^{d,W}$. Using this and right exactness of the tensor product, we show a few calculations. These include:

- Internal tensor product of functors of type Γ^λ over any commutative ring and a similar result for permutation modules for the symmetric group.
- A sketch of an algorithm to compute the internal tensor product of two Weyl functors over a field of characteristic zero. This algorithm becomes more practical when one of the involved partitions is fairly simple. We spell out the cases where one of the two partitions is a hook or has at most two rows.

*Chennai Mathematical Institute, Chennai, India. (upendra@cmi.ac.in).

†Chennai Mathematical Institute, Chennai, India. (shraddha@cmi.ac.in)

‡Chennai Mathematical Institute, Chennai, India. (kv@cmi.ac.in).

1 Introduction

Let k be a commutative ring and d be a positive integer. Let Γ_k^d denote the category of divided powers of degree d over k (see section 2.1 for the definition). The category of strict polynomial functors, denoted $\text{Rep}\Gamma_k^d$, is the category of k -linear representations of Γ_k^d (see section 2.2, for the definition).

The category of strict polynomial functors was first introduced by Friedlander and Suslin [FS97] in their work on the cohomology of group schemes. The description of polynomial functors given above is from Krause [Kra13]. As Krause mentions, this equivalent formulation of strict polynomial functors in terms of divided powers appears in the works of Kuhn [Kuh94], Pirashvili and Bousfield [Bou67].

It is well known that $\text{Rep}\Gamma_k^d$ is closely related to the category of modules over Schur algebras. In fact, by a result of Friedlander and Suslin [FS97], $\text{Rep}\Gamma_k^d$ is equivalent to the category of modules over the Schur algebra $S_k(n, d)$ when $n \geq d$. Schur algebras were introduced by Green [Gre80]. They are a link between the representations of general linear groups and symmetric groups. Denote by $\text{mod-}S_d$ the category of modules over the symmetric group on d letters. We have a functor called the Schur functor, which we denote Sch , from $\text{Rep}\Gamma_k^d$ to $\text{mod-}S_d$. The Schur functor is also discussed briefly in [Kra13, page10].

The starting point of our work is the definition of an internal tensor product in $\text{Rep}\Gamma_k^d$ by Krause [Kra13]. Our main result is that this internal product is related to the Kronecker product in $\text{mod-}S_d$, via the Schur functor. We prove

Theorem 1. *For X, Y strict polynomial functors of degree d , $\text{Sch}(X \otimes Y) = \text{Sch}(X) \otimes \text{Sch}(Y)$, where the action of S_d on the right hand side is given by the diagonal embedding of S_d in $S_d \times S_d$.*

In [Kra13] a crucial part is played the full subcategory of representable functors in $\text{Rep}\Gamma_k^d$. Our proof proceeds by examining the effect of the Schur functor on the representable functor $\Gamma^{d, W}$ (see 2.2 for the definition of $\Gamma^{d, W}$).

We also define and use strict polynomial functors X_λ^V that arise as weight spaces of a parametrized functor X^V . We compute the internal tensor product of a strict polynomial functor with $\Gamma_\lambda^{d, V}$. As a corollary, we compute the tensor product of permutation modules of the symmetric group for arbitrary k . In characteristic zero this corresponds to a well known result on complete symmetric functions (see [Sta01, Exercise 83, Chapter 7]).

We also calculate the internal tensor product of an arbitrary Weyl functor with Γ^λ . We get analogous results by applying Koszul duality and the Schur functor. We also give some examples of using internal tensor products to compute Kronecker coefficients in special cases in characteristic zero. This includes the case when one of the partition has two rows, and the case when one of the partition is a hook. Both these cases are well known in the literature.

As we were completing this paper, Aquilino and Reischuk [AR15] uploaded a paper on the arxiv, in which they too prove theorem 1. Their proof and approach is however different from ours.

Organization In section 2, we recall the definitions of the divided power category and strict polynomial functors. We then discuss the internal tensor product given in [Kra13], and state some propositions from that paper which we use. We discuss the Kuhn dual in $\text{Rep}\Gamma_k^d$, and external tensor products. We discuss Koszul duality, and state theorems from [Kra13] in the non-derived and derived worlds. In section 3, we give a very brief introduction to the representation theory of

the symmetric groups and define the Schur functor. In section 4, we first compute the image of $\Gamma^{d,W}$ under the Schur functor and use it to prove the main theorem. We then define polynomial functors X_λ^V which arise as weight spaces of the parametrized functor X^V , when V is a free module of finite rank. We use these functors to compute the internal tensor product of Γ^λ 's, solving a question posed by Krause. We then use this to compute internal tensor products of Weyl functors in special cases.

Our main reference will be [Kra13]. The notation we use is also taken from that paper. Sometimes we use techniques from [Kra13] without giving complete details.

Readers who are familiar with polynomial functors may skip directly to page 7.

2 Strict polynomial functors and internal tensor product

In this section we give the definition of strict polynomial functors and define the category $\text{Rep}\Gamma_k^d$. We also recall the internal tensor product on $\text{Rep}\Gamma_k^d$. Our references are the papers [Kra13] and [Tou13].

Let P_k denote the category of finitely generated projective modules over a commutative ring k . We note that P_k is closed under taking tensor products and k -linear hom's of modules over k . Hence it is a monoidal category with the usual hom-tensor adjunction. That is, we have a natural isomorphism

$$\text{Hom}_{P_k}(V \otimes W, U) \simeq \text{Hom}_{P_k}(V, \text{Hom}_k(W, U)).$$

Let M_k denote the category of modules over k and let $V \in M_k$. We define the divided power of V to be

$$\Gamma^d(V) := (V^{\otimes d})^{S_d}, \text{ where } S_d \text{ acts by permuting the factors of } V^{\otimes d}.$$

2.1 Divided power category

We define $\Gamma_{P_k}^d$, a category, whose objects are finitely generated projective modules over k . The morphisms between two objects V, W is defined to be

$$\text{Hom}_{\Gamma^d P_k}(V, W) := \Gamma^d(\text{Hom}_k(V, W))$$

We note that $\Gamma^d P_k$ inherits a monoidal structure from the monoidal structure of P_k .

2.2 Strict polynomial functor

A k -linear covariant functor from $\Gamma^d P_k$ to M_k is called a strict polynomial functor. The category of strict polynomial functors is denoted by $\text{Rep}\Gamma_k^d$. Note that being a functor category with an abelian target this is an abelian category.

A representation $X \in \text{Rep}\Gamma_k^d$ is by definition a pair of functions. The first assigns to each $X \in P_k$ a module $X(V)$ and the second assigns to each pair of objects $V, W \in P_k$ a k -linear map

$$X_{V,W} : \Gamma^d \text{Hom}(V, W) \rightarrow \text{Hom}(X(V), X(W)).$$

The functors $\otimes^d, \Gamma^d, \text{Sym}^d, \wedge^d$ belong to $\text{Rep}\Gamma_k^d$. The other functors in $\text{Rep}\Gamma_k^d$ which we will need are the Weyl functor $\Delta(\lambda)$ and the dual Weyl functor $\nabla(\lambda)$. A good reference for the last two functors is the article by Akin, Buchsbaum and Weyman [AB85].

An important class of strict polynomial functors was defined in Touzé [Tou13] and also in [Kra13]. These, *parametrized functors*, are defined using a polynomial functor X of degree d and a parametrization parameter, V , which is a finitely generated projective module.

$$X^V := X(\mathrm{Hom}_k(V, -))$$

Taking $X = \Gamma^{d,W}$ and using the Yoneda lemma, one sees that $\{\Gamma^{d,W}\}$ form a class of projective generators in $\mathrm{Rep}\Gamma_k^d$.

Remark 2. Touzé [Tou13] also defines the parametrized functor $X_V := X(V \otimes -)$.

As Krause observes, every strict polynomial functor X is the colimit of representable functors in $\mathrm{Rep}\Gamma_k^d$

$$X := \mathrm{colim}_{\Gamma^{d,V} \rightarrow X} \Gamma^{d,V}$$

Here the colimit is taken over the category of morphisms $\Gamma^{d,V} \mapsto X$, and V runs over objects in $\Gamma_{P_k}^d$.

2.3 Internal tensor product and Internal Homs

Krause defined an internal tensor product $\underline{\otimes}$ on $\mathrm{Rep}\Gamma_k^d$ [Kra13]. As Krause remarks, $\Gamma^d P_k$ being a monoidal category, this internal tensor product can be obtained via Day convolution. The internal hom on $\mathrm{Rep}\Gamma_k^d$, denoted $\mathbb{H}_{\mathrm{Rep}\Gamma_k^d}(-, -)$ was defined by Touzé [Tou13]. We recall proposition [Kra13, Proposition 2.4]

Proposition 3. *The bifunctors $- \otimes -$ and $\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(-, -)$ for $\Gamma^d P_k$ induce, via the Yoneda embedding bifunctors*

$$\begin{aligned} - \underline{\otimes} - & : - \quad \mathrm{Rep}\Gamma_k^d \times \mathrm{Rep}\Gamma_k^d \rightarrow \mathrm{Rep}\Gamma_k^d \\ \mathbb{H}_{\mathrm{Rep}\Gamma_k^d}(-, -) & : - \quad (\mathrm{Rep}\Gamma_k^d)^{op} \times \mathrm{Rep}\Gamma_k^d \rightarrow \mathrm{Rep}\Gamma_k^d \end{aligned}$$

with a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(X \underline{\otimes} Y, Z) \simeq \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(X, \mathbb{H}_{\mathrm{Rep}\Gamma_k^d}(Y, Z))$$

One defines this by requiring for $V, W \in \Gamma^d P_k$ that

$$\begin{aligned} \Gamma^{d,V} \underline{\otimes} \Gamma^{d,W} & = \Gamma^{d,V \otimes W} \\ \mathbb{H}_{\mathrm{Rep}\Gamma_k^d}(V, W) & = \Gamma^{d, \mathrm{Hom}_k(V, W)} \end{aligned}$$

and this determines both the bifunctors, using the fact every polynomial functor is the colimit of representable functors.

This internal tensor product is right exact being a left adjoint functor.

2.4 Duality

As Krause observes, $\text{Rep}\Gamma_k^d$ has a duality theory. This is the Kuhn dual, and is defined as follows: For any polynomial functor X , the Kuhn dual X° is defined as

$$X^\circ(V) := X(V^*)^*$$

The Kuhn dual of a parametrized functor satisfies

$$(X^V)^\circ = ((X)^\circ)^{V^*} = (X^\circ)_V,$$

In particular $(\Gamma^{d,V})^\circ = (\text{Sym}^{d,V^*})$ and $(\wedge^{d,V})^\circ = \wedge^{d,V^*}$.

For several other properties of internal tensor product and internal hom we refer the reader to [Kra13] and [Tou13].

2.5 The external tensor product

Given two polynomial functors X, Y of degree d, e respectively, Krause defines their external tensor product $X \otimes Y$, of degree $d + e$. On objects $X \otimes Y(V) := X(V) \otimes Y(V)$. To describe the morphisms between objects we refer to [Kra13]. We need some more notation to describe the various polynomial functors we use in the rest of the paper. All these functors were defined in [Kra13]. We also discuss some results from [Kra13] which we need.

For integers $d \geq 0$ and $n \geq 0$, we denote by $\Lambda(n, d)$, the set of sequences $\lambda = (\lambda_1 \dots \lambda_n)$ of non-negative integers such that $\sum_{i=1}^n \lambda_i = d$.

Given $\lambda \in \Lambda(n, d)$, we set

$$\begin{aligned} \Gamma^\lambda &:= \Gamma^{\lambda_1} \otimes \dots \otimes \Gamma^{\lambda_n} \\ \text{Sym}^\lambda &:= \text{Sym}^{\lambda_1} \otimes \dots \otimes \text{Sym}^{\lambda_n} \\ \wedge^\lambda &:= \wedge^{\lambda_1} \otimes \dots \otimes \wedge^{\lambda_n} \end{aligned}$$

As noted in [Kra13], the divided power functor, the symmetric power functor and the exterior power functor all have the exponential property. That is, we have graded k -algebra homomorphisms

$$\begin{aligned} \Gamma(V \oplus W) &\simeq \Gamma(V) \otimes \Gamma(W) \\ \text{Sym}(V \oplus W) &\simeq \text{Sym}(V) \otimes \text{Sym}(W) \\ \wedge(V \oplus W) &\simeq \wedge(V) \otimes \wedge(W) \end{aligned}$$

In particular, using the exponential property of the divided powers Krause showed [Kra13, 2.8]

$$\Gamma^{d,V} = \bigoplus_{\lambda \in \Lambda(\dim V, d)} \Gamma^\lambda$$

So each Γ^λ is a summand of a projective generator.

2.6 Connections to the Schur Algebra

Using the functoriality of a polynomial functor X of degree d , there is an algebra homomorphism,

$$\Gamma^d \text{Hom}(k^n, k^n) \rightarrow \text{Hom}(X(k^n), X(k^n)).$$

Thus evaluating a polynomial functor at k^n gives an action of the algebra $\Gamma^d \text{End}(k^n)$ on $X(k^n)$. The algebra $\Gamma^d \text{End}(k^n)$ is isomorphic to the Schur algebra $S_k(n, d)$. So ev_{k^n} is a functor from $\text{Rep}\Gamma_k^d$ to $\text{mod-}S(n, d)$. In fact this is an equivalence of categories when $n \geq d$. This was proved by Suslin and Friedlander [FS97]. We state this for future reference.

Proposition 4. [FS97] Let d, n be positive integers with $n \geq d$. Then the evaluation functor $ev_k^n : \text{Rep}\Gamma_k^d \rightarrow \text{mod } -S_k(n, d)$ is equivalence of categories.

Remark 5. It is also known that the category of polynomial representations of $GL(n)$ of degree d is equivalent to the category $\text{mod } -S_k(n, d)$ [Gre80].

2.7 Koszul duality

Classical Koszul duality deals with the homological relations between symmetric and exterior powers, namely

$$\begin{aligned} \text{Ext}_{\text{Sym}(V^*)}(k, k) &\simeq \wedge V \\ \text{Ext}_{\wedge(V^*)}(k, k) &\simeq \text{Sym}(V) \end{aligned}$$

Here k is the trivial module for both $\text{Sym}(V^*)$ and $\wedge(V^*)$. More details can be found in [Kra13, Sec 4.7]. For strict polynomial functors, Koszul duality has been discussed by Chałupnik and Touzé [Cha08] [Tou13]. We recall Koszul duality, and discuss a few facts and results about Koszul duality from [Kra13].

Lemma 6. [Kra13, Lemma 3.5] Given $d \geq 1$ and $V \in P_k$, there are exact sequences

$$\begin{aligned} \bigoplus_{i=1}^{d-1} V^{\otimes i-1} \otimes \Gamma^2 V \otimes V^{d-i-1} \xrightarrow{1 \otimes \Delta \otimes 1} V^{\otimes d} \rightarrow \wedge^d(V) \rightarrow 0 \\ \bigoplus_{i=1}^{d-1} V^{\otimes i-1} \otimes \wedge^2 V \otimes V^{d-i-1} \xrightarrow{1 \otimes \Delta \otimes 1} V^{\otimes d} \rightarrow \text{Sym}^d(V) \rightarrow 0 \end{aligned}$$

where $\Delta : \Gamma^2 V \rightarrow V \otimes V$ is the component of the comultiplication which is dual to the multiplication $V^* \otimes V^* \rightarrow \text{Sym}^2(V^*)$, and $\Delta : \wedge^2 \rightarrow V \otimes V$ is the component of the comultiplication given by $\Delta(v \wedge w) = v \otimes w - w \otimes v$.

Proposition 7. [Kra13, Prop 3.6], [Tou13, Lemma 3.6] Let $d \geq 1$. Then

$$\wedge^d \underline{\otimes} \wedge^d \simeq \text{Sym}^d$$

Corollary 8. [Kra13, Corollary 3.7] Let $\lambda \in \Lambda(n, d)$. Then there is an isomorphism

$$\wedge^d \underline{\otimes} \wedge^\lambda \simeq \text{Sym}^\lambda.$$

For a set of objects C of an additive category, we denote by $\text{add}C$ the full subcategory consisting of all finite direct sums in C and their direct summands. Krause shows,

Corollary 9. [Kra13, Corollary 3.8] Let $d, n \geq 1$. The functor $\wedge^d \underline{\otimes} -$ induces equivalences

$$\begin{aligned} \text{add}\{\Gamma^\lambda \mid \lambda \in \Lambda(n, d)\} &\xrightarrow{\sim} \text{add}\{\wedge^\lambda \mid \lambda \in \Lambda(n, d)\} \\ \text{add}\{\wedge^\lambda \mid \lambda \in \Lambda(n, d)\} &\xrightarrow{\sim} \text{add}\{\text{Sym}^\lambda \mid \lambda \in \Lambda(n, d)\} \end{aligned}$$

with quasi-inverses induced by $\mathbb{H}(\wedge^d, -)$.

The non-derived version of Koszul duality given above, by tensoring with $\wedge^d \underline{\otimes} -$, is an equivalence of two subcategories in $\text{Rep}\Gamma_k^d$.

On the other hand, the derived version induces an equivalence of the unbounded derived category of $\text{Rep}\Gamma_k^d$ with itself, as we shall recall in the next section.

2.8 Derived Koszul duality

Since the internal tensor product with either one of the argument fixed, is a right exact functor, we take its left derived functors. We briefly discuss some of results about derived Koszul duality $\wedge^d \underline{\otimes}^{\mathbf{L}} -$ from [Kra13, Section 4], [Tou13], which will be used in the section 5.

Theorem 10. *The functors $\wedge^d \underline{\otimes}^{\mathbf{L}} -$ and $\mathbf{R}\mathcal{H}om_{\Gamma_k^d}(\wedge^d, -)$, provide mutually quasi-inverse equivalences*

$$D(\text{Rep}) \Gamma_k^d \xrightarrow{\sim} D(\text{Rep}) \Gamma_k^d$$

To compute $\wedge^d \underline{\otimes}^{\mathbf{L}} \Delta(\lambda)$ one starts with a projective resolution of $\Delta(\lambda)$. Such a resolution in terms of divided powers is known from the work of Akin, Buchsbaum and Weyman [AB85]. Applying $\wedge^d \underline{\otimes}^{\mathbf{L}} -$ we obtain the resolution of $\nabla(\lambda)$.

Proposition 11. *Let λ be a partition of d . Then we an isomorphism,*

$$\wedge^d \underline{\otimes}^{\mathbf{L}} \Delta(\lambda) \simeq \nabla(\lambda'),$$

where λ' is the transpose partition of λ .

We remark that the higher left derived functors of the Koszul dual functor on Weyl functors vanishes that is

$$L^i(\wedge^d \underline{\otimes}^{\mathbf{L}} \Delta(\lambda)) = 0, \quad i \geq 1$$

3 Representations of the symmetric group and the Schur functor

3.1 Symmetric group representations

We give a very short introduction to the representations of symmetric group S_d . We only define those terms which will be required in the later part of this paper. Standard references for this are the book by Martin [Mar09, Chapter 4], Sagan [Sag01] and the book by James [Jam78].

It is very well-known that the representations of the S_d are intimately linked to the partitions of d . These are in turn in one-one correspondence with the Young diagrams of d . We do not recollect these notions for the reader, instead referring to the books cited earlier.

A tabloid is an equivalence class of labellings of the Young diagrams of shape $\lambda := (\lambda_1, \dots, \lambda_n)$ of d . Two labellings are said to be equivalent if one can obtained from the other by permuting the entries in each row.

Denote by $\{T\}$ the equivalence class of a tableau T . The symmetric group S_d acts on the set of tableaux of shape λ (i.e., on the set of labellings of the Young diagram). Consequently, it acts on tabloids.

Definition 12. *The S_d module with tabloids of shape λ as basis is called the permutation module, and we denote it by M^λ*

Remark 13. These tabloids also can be thought of as functions $f : [d] := \{1, \dots, d\} \rightarrow [n]$. Here $|f^{-1}(i)| = \lambda_i$ for $1 \leq i \leq n$. And the elements mapping to i are precisely the labels in the i -th. Now S_d acts on the space of functions by permuting the elements in the domain $[d]$. It is easy to see that this gives us the same S_d module M^λ . It is also easy to see that the permutation module is induced from the trivial representation of the subgroup $S_\lambda := S_{\lambda_1} \times \dots \times S_{\lambda_n}$, i.e. $M^\lambda := \text{Ind}_{S_\lambda}^{S_d} k1$. S_λ is called the Young subgroup of S_d corresponding to shape λ .

For each Young tableau T of shape λ , we form the element $\kappa_T = \sum_{\pi \in C_T} (\text{sgn } \pi)\pi$, where C_T is the column stabilizer of the Young tableau T

The polytabloid, e_T , associated with the tableau T is given defined to be $e_T = \{T\}\kappa_T$.

Definition 14. *The Specht module S^λ corresponding to the partition λ is the submodule of the permutation module M^λ spanned by the polytabloids.*

3.2 Schur functor

We recall the definition of the Schur functor as defined by Krause [Kra13, Theorem 2.10]. It is defined as $\text{Hom}_{\text{Rep}\Gamma_k^d}(\otimes^d, -) : \text{Rep}\Gamma_k^d \rightarrow \text{mod-}S_d$. We use the notation Sch for this functor, to emphasize the connection between modules over the Schur algebra $S(n, d)$ and the symmetric group S_d outlined earlier in section 2.6. The fact that the left hand side acquires a S_d structure, follows from the theorem $\text{Hom}(\otimes^d, \otimes^d) \simeq kS_d$. A proof of this is given in [Tou12].

It is well-known that Sch is an equivalence of categories. We provide a short proof.

Proposition 15. *Sch is an equivalence of categories when $\text{char } k = 0$ or $\text{char } k = p > d$.*

Proof. For $\text{char } k = 0$ or $\text{char } k = p > d$, and for $\lambda \in \Lambda(d, d)$, we have an exact sequence

$$\otimes^d \rightarrow \Gamma^\lambda \rightarrow 0,$$

Using the exponential property of Γ^{d, k^d} , and the fact that it is a small projective generator [Kra13, Section 2] we conclude that \otimes^d is also a small projective generator in $\text{Rep}\Gamma_k^d$. Thus Sch is an equivalence of categories. \square

This functor is well studied. In fact from [Mar09], and [CHN10], it is well known that $\text{Sch}(\nabla(\lambda)) = S^\lambda$, the Specht module and that $\text{Sch}(\Delta(\lambda)) = (S^\lambda)^*$, the dual Specht module. We also recall a well known result which we will need. A characteristic free proof of this can be found in the paper by Santana and Yudin [SY12].

Proposition 16. *Let $\lambda \in \Lambda(n, d)$. Then*

$$\text{Sch}(\Gamma^\lambda) = M^\lambda.$$

4 Main result

Next we show that Sch takes the internal tensor product on $\text{Rep}\Gamma_K^d$ to the Kronecker product on $\text{mod-}S_d$. For this we first find $\text{Sch}(\Gamma^{d, V})$

Proposition 17. *$\text{Sch}(\Gamma_V^d) = V^{\otimes d}$ with S_d action by just permuting the factors of V .*

Proof. Let e_1, e_2, \dots, e_d be a basis of the vector space k^d . We identify $V \otimes k^d$ with $V_1 \oplus V_2 \oplus \dots \oplus V_d$ where $V_i = V \otimes ke_i \cong V$. Using this identification, and the definition of the Schur functor we have

$$\begin{aligned} \text{Sch}(\Gamma_V^d) &= \text{Hom}_{\text{Rep}\Gamma_k^d}(\otimes^d, \Gamma_V^d) \\ &= [\Gamma_V^d(k^d)]_{1^d} \\ &= [\Gamma^d(V \otimes k^d)]_{1^d} \\ &= [\Gamma^d(V_1 \oplus V_2 \oplus \dots \oplus V_d)]_{1^d} \end{aligned}$$

Using the exponential property of Γ^d we get

$$\begin{aligned} [\Gamma^d(V_1 \oplus V_2 \oplus \dots \oplus V_d)]_{1^d} &= \bigoplus_{\lambda \in \Lambda(d,d)} [\Gamma^{\lambda_1}(V_1) \otimes \dots \otimes \Gamma^{\lambda_d}(V_d)]_{1^d} \\ &= V_1 \otimes \dots \otimes V_d \\ &\cong V^{\otimes d} \end{aligned}$$

The second equality follows since $\Gamma^{\lambda_1}(V_1) \otimes \Gamma^{\lambda_2}(V_2) \dots \otimes \Gamma^{\lambda_d}(V_d)$ is the $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ -weight space of $\Gamma_V^d(k^d)$, and we are taking the weight space $1^d = (1, 1, \dots, 1)$. The last line follows since $V_i \cong V$, for all i . S_d acts on $V^{\otimes d}$ by permuting the tensor factors. \square

Using the definition of the Kuhn dual of a parametrized functor from section 2.4, and the above proposition it follows

Corollary 18. $\text{Sch}(\Gamma^{d,V}) = (V^*)^{\otimes d}$

Remark 19. It also follows from the above proposition that $\text{Sch}(X^\circ) = \text{Sch}(X)^*$, where $*$ denotes the dual in the category $\text{mod-}S_d$.

We state the main theorem again for completeness, and give the proof.

Theorem 20. *For X, Y strict polynomial functors of degree d , $\text{Sch}(X \underline{\otimes} Y) = \text{Sch}(X) \otimes \text{Sch}(Y)$, where the action of S_d on the right hand side is by the diagonal embedding in $S_d \times S_d$.*

Proof. We prove the proposition first in the case when $X = \Gamma^{d,V}$ and $Y = \Gamma^{d,W}$ are representable functors. From proposition 17 we have

$$\begin{aligned} \text{Sch}(\Gamma^{d,V} \underline{\otimes} \Gamma^{d,W}) &= \text{Sch}(\Gamma^{d,V \otimes W}) \\ &= ((V \otimes W)^*)^{\otimes d} \\ &\cong (V^*)^{\otimes d} \otimes (W^*)^{\otimes d} \\ &= \text{Sch}(\Gamma^{d,V}) \otimes \text{Sch}(\Gamma^{d,W}) \end{aligned}$$

For general X and Y we follow the technique from [Kra13] of expressing X, Y as a colimit of the representable functors. We use the fact that Sch , $\underline{\otimes}$ and Kronecker \otimes product preserve colimits, since they are left adjoints. First assume that Y is representable.

$$\begin{aligned} \text{Sch}(X \underline{\otimes} \Gamma^{d,W}) &= \text{Sch}\left(\text{colim}_{\Gamma^{d,U} \rightarrow X} \Gamma^{d,U} \underline{\otimes} \Gamma^{d,W}\right) \\ &= \text{Sch}\left(\text{colim}_{\Gamma^{d,U} \rightarrow X} \Gamma^{d,U \otimes W}\right) \\ &= \text{colim}_{\Gamma^{d,U} \rightarrow X} \text{Sch}(\Gamma^{d,U \otimes W}) \\ &= \text{colim}_{\Gamma^{d,U} \rightarrow X} \text{Sch}(\Gamma^{d,U} \underline{\otimes} \Gamma^{d,W}) \\ &= \text{colim}_{\Gamma^{d,U} \rightarrow X} [\text{Sch}(\Gamma^{d,U}) \otimes \text{Sch}(\Gamma^{d,W})] \\ &= \text{Sch}\left(\text{colim}_{\Gamma^{d,U} \rightarrow X} \Gamma^{d,U}\right) \otimes \text{Sch}(\Gamma^{d,W}) \\ &= \text{Sch}(X) \otimes \text{Sch}(\Gamma^{d,W}) \end{aligned}$$

To complete the proof, use the same technique for general Y . □

4.1 Weight spaces of parametrized functors

In this section we compute the internal tensor product $X \underline{\otimes} \Gamma^\lambda$. In the case when $X = \Gamma^\mu$ we get an explicit expression for $\Gamma^\mu \underline{\otimes} \Gamma^\lambda$, answering a question posed by Krause.

It will be useful to define the notion of the weight space of a parametrized functor.

Definition 21. *Let V be a free module over k of finite rank n . Let X be a homogeneous polynomial functor of degree d . Let $\lambda \in \Lambda(n, d)$ be a composition of d . For every finitely generated projective module U over k we have an algebra map*

$$\Gamma^d \text{Hom}(V^*, V^*) \rightarrow \text{End}(X(V^* \otimes U))$$

and via this map $X(V^* \otimes U)$ is a polynomial representation of $GL(V^*)$ of degree d . Define

$$X(V^* \otimes U)_\lambda := \{a \in X(V^* \otimes U) \mid X(t \otimes (id_U)^{\otimes d}).a = t_1^{\lambda_1} \dots t_n^{\lambda_n}.a \ \forall t \in \mathbb{G}_m^n\}$$

We check below that this is functorial and gives us a strict polynomial functor. We define $X_\lambda^V(-) := X(V^* \otimes -)_\lambda$. This parametrized weight functor is thought of as the λ -weight space of the parametrized functor $X^V(-) := X \circ \text{Hom}(V, -)$ defined in [Kra13, Lemma 2.5].

We check functoriality. Since X is a polynomial functor we have a k -linear map

$$\Gamma^d \text{Hom}(U, W) \rightarrow \text{Hom}_k(X(U \otimes V^*), X(W \otimes V^*))$$

The commutative diagram

$$\begin{array}{ccc} X(U \otimes V^*) & \xrightarrow{X(\phi \otimes id_{V^*}^{\otimes d})} & X(W \otimes V^*) \\ \downarrow X(id_U^{\otimes d} \otimes f) & & \downarrow X(id_W^{\otimes d} \otimes f) \\ X(U \otimes V^*) & \xrightarrow{X(\phi \otimes id_{V^*}^{\otimes d})} & X(W \otimes V^*) \end{array}$$

shows that we have a map

$$\Gamma^d \text{Hom}(U, W) \rightarrow \text{Hom}_k(X(U \otimes V^*)_\lambda, X(W \otimes V^*)_\lambda).$$

We show how the notion of weight space is related to internal tensor product.

Lemma 22. *Let W be a free module of finite rank. Then*

$$X_\lambda^W = X \underline{\otimes} \Gamma_\lambda^{d, W}.$$

Proof. We first show this in the case $X = \Gamma^{d, V}$. Applied to U , the left hand side is the λ -weight space of $\text{Hom}(W^* \otimes V^* \otimes U)^{S_d}$. By definition, the right hand side is $\Gamma_\lambda^{d, W}(\text{Hom}(V, U)) = \Gamma_\lambda^{d, W}(V^* \otimes U)$. Now this is precisely the λ -weight space of $\text{Hom}(W^* \otimes V^* \otimes U)^{S_d}$.

If $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is an exact sequence in $\text{Rep}\Gamma_k^d$, then $0 \rightarrow X_\lambda^W \rightarrow Y_\lambda^W \rightarrow Z_\lambda^W \rightarrow 0$ is also exact. Now let $X = \text{colim}_{\Gamma^{d,V} \rightarrow X} \Gamma^{d,V}$. Then we have

$$\begin{aligned} \left(\text{colim}_{\Gamma^{d,V} \rightarrow X} \Gamma^{d,V} \right) \underline{\otimes} \Gamma_\lambda^{d,W} &= \text{colim}_{\Gamma^{d,V} \rightarrow X} (\Gamma^{d,V} \underline{\otimes} \Gamma_\lambda^{d,W}) \\ &= \text{colim}_{\Gamma^{d,V} \rightarrow X} (\Gamma^{d,V})_\lambda^W \\ &= \left(\text{colim}_{\Gamma^{d,V} \rightarrow X} \Gamma^{d,V} \right)_\lambda^W \\ &= X_\lambda^W \end{aligned}$$

The first line follows since internal tensor product commutes with colimits. The second line was proved in the paragraph above. The functor $Y \rightarrow Y_\lambda^W$ preserves colimits since it is exact and preserves arbitrary direct sums in $\text{Rep}\Gamma_k^d$, which is an abelian category. \square

Let μ be a partition of d into n parts and λ a partition of d into m parts. It is clear from the above definition that we may identify the strict polynomial functor Γ^μ with Γ_μ^{d,k^n} and the functor Γ^λ with Γ_λ^{d,k^m} .

Proposition 23. $\Gamma^\mu \underline{\otimes} \Gamma^\lambda = \bigoplus_{\nu \in S} \Gamma^\nu$, where ν is a composition of d with at most nm parts, obtained naturally from the set S of n by m matrices with non-negative integer entries, with row sums μ and column sums λ .

Proof. Let $U \in \Gamma^d P_k$. Then

$$\begin{aligned} \Gamma^\mu \underline{\otimes} \Gamma^\lambda(U) &= (\Gamma_\mu^{d,k^n} \underline{\otimes} \Gamma_\lambda^{d,k^m})(U) \\ &= (\Gamma_\mu^{d,k^n} (k^{m*} \otimes U))_\lambda \\ &= (\Gamma^d(k^{n*} \otimes k^{m*} \otimes U)_\mu)_\lambda \end{aligned}$$

In the above, the λ -weight space is taken with respect to $GL(k^{m*})$ and the μ -weight space is taken with respect to $GL(k^{n*})$. Thus, $\Gamma^\mu \underline{\otimes} \Gamma^\lambda(U)$ is the (μ, λ) weight space of $\Gamma^{d,k^n \otimes k^m}(U)$ with respect to the action of $GL(k^{n*}) \times GL(k^{m*})$.

On the other hand $\Gamma^{d,k^n \otimes k^m}(U)$ is a polynomial representation of $GL(k^{n*} \otimes k^{m*})$ whose weights are given by $n \times m$ non-negative integer matrices (ν_{ij}) with the entries add up to d . This induces a weight space decomposition of $\Gamma^{d,k^n \otimes k^m}(U)$ for the restriction of the action to

$$\begin{aligned} GL(k^{n*}) \times GL(k^{m*}) &\hookrightarrow GL(k^{n*} \otimes k^{m*}) \\ A \times B &\mapsto A \otimes B . \end{aligned}$$

Under the restriction to $GL(k^{n*}) \times GL(k^{m*}) \hookrightarrow GL(k^{n*} \otimes k^{m*})$ elements of the (ν_{ij}) -weight space have $GL(k^{n*}) \times GL(k^{m*})$ weight $(\mu_1, \dots, \mu_n, \lambda_1, \dots, \lambda_m)$, where $\mu_i = \sum_{j=1}^m \nu_{ij}$, and $\beta_j = \sum_{i=1}^n \nu_{ij}$. \square

We apply Sch to the above and use proposition 16 to obtain analogous description of the Kronecker product of two permutation modules, which is well known in characteristic zero [Mac95].

Corollary 24. $M^\mu \otimes M^\lambda = \bigoplus_{\nu \in S} M^\nu$ where S is the set of n by m matrices with non-negative entries with row sums μ and column sums λ .

5 Examples and applications to Kronecker multiplicities

In this section we calculate the internal tensor product of an arbitrary Weyl functor with summands Γ^ν of the projective generator Γ^{d,k^n} . Application of Koszul duality and of the Schur functor give parallel results. We also explore implications of our calculation for the classical Kronecker problem of decomposing tensor products of Specht modules. Here is a summary of our results.

1. We show that $\Delta(\lambda)\underline{\otimes}\Gamma^\nu$ has an explicit Weyl filtration that is independent of the ground ring k . Applying Koszul duality and the Schur functor, we deduce that the resulting objects have dual Weyl and (dual) Specht filtrations. The multiplicities in these filtrations are given by sums of products of Littlewood-Richardson coefficients. By contrast, the internal tensor product of two Weyl functors does not have a Weyl filtration in general, e.g., $\wedge^d \underline{\otimes} \wedge^d \simeq \text{Sym}^d$ by [Kra13]. As a small sample of dependence of $\underline{\otimes}$ on the ground ring, we observe that $\text{Sym}^d \underline{\otimes} \wedge^d \simeq \wedge^d$ if 2 is invertible in k , but different if $2 = 0$. See [Tou13] for deeper exploration of “Ringel duality” in terms of calculations of derived internal Hom.
2. Let k be a field of characteristic 0. Then the Schur functor is an equivalence between semisimple categories. One may wonder if Theorem 1 can be used to study the well known Kronecker problem, which asks for a good description of multiplicities of Specht modules in the tensor product of two Specht modules. While our calculations do yield an algorithm to decompose the internal tensor product $\Delta(\lambda)\underline{\otimes}\Delta(\mu)$, all the ingredients of the (fairly involved) procedure have known parallels for the internal product of Schur functions [Mac95, section 6, example 23C]. We still find it interesting to spell everything out when one of λ and μ is either a two-row partition or a hook, as here the resulting procedure is reasonably simple. In case of a hook, this uses a signed version of polynomial functors defined by Axtell [J13].

Description of $\Delta(\lambda)\underline{\otimes}\Gamma^\nu$. Let k be an arbitrary commutative ring and let λ and μ be partitions with $\mu \subset \lambda$. In [AB85], Akin and Buchsbaum gives an explicit construction of a filtration of $\Delta(\lambda/\mu)(V \oplus W)$ that is universal (i.e., independent of the ground ring k), functorial in V and W and whose associated graded object is

$$\bigoplus_{\substack{\text{partitions } \alpha \\ \mu \subset \alpha \subset \lambda}} \Delta(\alpha/\mu)(V) \otimes \Delta(\lambda/\alpha)(W).$$

Apply this repeatedly to $\Delta(\lambda/\mu)^{k^n}(V) \simeq \Delta(\lambda/\mu)(V \oplus V \oplus \dots \oplus V)$. We get a filtration whose associated graded object is

$$\bigoplus_{\substack{\text{partitions } \alpha^1, \alpha^2, \dots, \alpha^{n-1} \\ \mu \subset \alpha^1 \subset \alpha^2 \subset \dots \subset \alpha^{n-1} \subset \lambda}} \Delta(\alpha^1/\mu)(V) \otimes \Delta(\alpha^2/\alpha^1)(V) \otimes \dots \otimes \Delta(\lambda/\alpha^{n-1})(V). \quad (1)$$

Now let $\nu = (\nu_1, \dots, \nu_n) \in \Lambda(n, d)$. By lemma 22,

$$\Delta(\lambda/\mu)\underline{\otimes}\Gamma^\nu(V) \simeq \Delta(\lambda/\mu)_\nu^{k^n}(V) \simeq \Delta(\lambda/\mu)(\text{Hom}(k^n, V))_\nu \simeq \Delta(\lambda/\mu)(V \oplus V \oplus \dots \oplus V)_\nu.$$

Taking the ν weight space is equivalent to requiring $|\alpha^i/\alpha^{i-1}| = \nu_i$ for all $i = 1, \dots, n$ in equation 1, with the understanding that $\alpha^0 = \mu$ and $\alpha^n = \lambda$.

Altogether, we get a filtration of $\Delta(\lambda/\mu)\underline{\otimes}\Gamma^\nu$ whose associated graded object is

$$\bigoplus_{\substack{\text{partitions } \alpha^0 \subset \alpha^1 \subset \alpha^2 \subset \dots \subset \alpha^{n-1} \subset \alpha^n \\ \alpha^0 = \mu, \alpha^n = \lambda, |\alpha^i / \alpha^{i-1}| = \nu_i}} \Delta(\alpha^1 / \mu) \otimes \Delta(\alpha^2 / \alpha^1) \otimes \dots \otimes \Delta(\lambda / \alpha^{n-1}).$$

Finally note that each tensor product in the above expression itself has a Weyl filtration in which multiplicity of any $\Delta(\beta)$ can be calculated as a sum of products of Littlewood-Richardson coefficients. This is because $\Delta(\alpha^i / \alpha^{i-1})$ has a Weyl filtration in which multiplicity of $\Delta(\beta)$ is given by the Littlewood-Richardson coefficient $c_{\alpha^{i-1}\beta}^{\alpha^i}$. Applying this in all tensor slots leads to a filtration whose successive quotients are n -fold tensor products of Weyl functors. These in turn have Weyl filtrations with multiplicities given by products of Littlewood-Richardson coefficients. We summarize part of the above discussion in the following qualitative statement.

Proposition 25. $\Delta(\lambda) \underline{\otimes} \Gamma^\nu$ has a Weyl filtration that is independent of the ground ring k and in which multiplicity of any Weyl functor can be calculated as a sum of products of Littlewood-Richardson coefficients.

The property of having a Weyl filtration passes to direct summands. As any projective object in the $\text{Rep}\Gamma_k^d$ is a summand of a direct sum of various Γ^ν , we get

Corollary 26. The internal tensor product of a Weyl functor and a finitely generated projective object in $\text{Rep}\Gamma_k^d$ has a Weyl filtration.

Corollary 27. $\Delta(\lambda) \underline{\otimes} \wedge^\nu \simeq \nabla(\lambda') \underline{\otimes} \Gamma^\nu$ has a dual Weyl filtration that is independent of the ground ring k and in which multiplicity of any dual Weyl functor can be calculated as a sum of products of Littlewood-Richardson coefficients.

Proof. Apply $-\underline{\otimes} \wedge^d$ to proposition 25, where $\underline{\otimes}$ is the derived version of \otimes . By [Kra13], we have $\wedge^d \underline{\otimes} \Gamma^\nu \simeq \wedge^\nu$ and $\wedge^d \underline{\otimes} \Delta(\lambda) \simeq \nabla(\lambda')$. In particular, functors with a Weyl filtration are acyclic for the left exact functor $\wedge^d \underline{\otimes} -$. We deduce that

$$\Delta(\lambda) \underline{\otimes} \wedge^\nu \simeq \Delta(\lambda) \underline{\otimes} \wedge^d \underline{\otimes} \Gamma^\nu \simeq \nabla(\lambda') \underline{\otimes} \Gamma^\nu$$

and that applying $\wedge^d \underline{\otimes} -$ turns a Weyl filtration into a dual Weyl filtration. Note also that this and all prior results in this section remain valid after replacing $\underline{\otimes}$ with $\underline{\otimes}^L$. □

Remark: Further applications of $\wedge^d \underline{\otimes} -$ lead to objects that depend on the ground ring k . For example, consider the right exact sequence

$$\bigoplus_{i=0}^{d-2} \left(\otimes^i \otimes \wedge^2 \otimes \otimes^{d-i-2} \right) \rightarrow \otimes^d \rightarrow \text{Sym}^d \rightarrow 0.$$

Applying $\wedge^d \underline{\otimes} -$ gives

$$\bigoplus_{i=0}^{d-2} \left(\otimes^i \otimes \text{Sym}^2 \otimes \otimes^{d-i-2} \right) \rightarrow \otimes^d \rightarrow \wedge^d \underline{\otimes} \text{Sym}^d \rightarrow 0.$$

If 2 is invertible in k , then $\text{Sym}^2 \simeq \Gamma^2$. Substituting and appealing to [Tou13] to identify the leftmost map, we see that in this case its cokernel $\wedge^d \underline{\otimes} \text{Sym}^d \simeq \wedge^d$. However, if $2 = 0$ in k , the cokernel is clearly bigger.

Application to the symmetric group and the Kronecker problem. Applying the Schur functor to Proposition 25 and Corollary 27 we obtain

Corollary 28. *The Kronecker product of a (dual) Specht module with a permutation module has an explicitly constructed (dual) Specht filtration that is independent of the ground ring k .*

For the rest of this section, let k be a field of characteristic 0. Now Sch is an equivalence of semisimple categories and $\underline{\otimes}$ is exact. For partitions λ and μ of d , we can use equation 1 to obtain a procedure to calculate $\Delta(\lambda) \underline{\otimes} \Delta(\mu)$. For example, the Jacobi-Trudi formula expresses $\Delta(\mu)$ as an alternating sum of tensor products of various Γ^ν and now we can use equation 1 repeatedly. While the resulting algorithm is quite laborious in general, we will see below that when μ is a two-row partition, the procedure is reasonably simple. When μ is a hook a comparably simple variation can be devised.

(i) We calculate $\Delta(\lambda) \underline{\otimes} \Delta((a, b))$, where $a \geq b$ are positive integers with $a + b = d$. We have

$$\begin{aligned} \Delta(\lambda) \underline{\otimes} \Gamma^{(a,b)} &\simeq \Delta(\lambda)_{(a,b)}^{k^2} \\ &\simeq \bigoplus_{\mu \subset \lambda, |\mu|=a} \Delta(\mu) \otimes \Delta(\lambda/\mu) \\ &\simeq \bigoplus_{\mu \subset \lambda, |\mu|=a} \Delta(\mu) \otimes \left(\bigoplus_{\nu \subset \lambda, |\nu|=b} c_{\mu,\nu}^\lambda \Delta(\nu) \right) \\ &\simeq \bigoplus_{|\mu|=a, |\nu|=b, |\alpha|=d} c_{\mu,\nu}^\lambda c_{\mu,\nu}^\alpha \Delta(\alpha) \end{aligned}$$

We also have $\Gamma^{(a,b)} \simeq \Gamma^{(a+1,b-1)} \oplus \Delta((a, b))$ by, e.g., Pieri's formula. Apply $-\underline{\otimes} \Delta(\lambda)$ and use the above calculation to get

$$\Delta(\lambda) \underline{\otimes} \Delta((a, b)) \simeq \left(\sum_{|\mu|=a, |\nu|=b} c_{\mu,\nu}^\lambda c_{\mu,\nu}^\alpha - \sum_{|\bar{\mu}|=a+1, |\bar{\nu}|=b-1} c_{\bar{\mu},\bar{\nu}}^\lambda c_{\bar{\mu},\bar{\nu}}^\alpha \right) \Delta(\alpha)$$

As a special case consider $b = 1$ and let $c =$ the number of outer corners of λ . Now $\Gamma^{(a+1,b-1)} = \Gamma^d$, which is the identity for $\underline{\otimes}$. We get

$$\Delta(\lambda) \underline{\otimes} \Delta((a, 1)) \simeq (c - 1) \Delta(\lambda) \bigoplus_{\alpha} \Delta(\alpha),$$

where α ranges over partitions obtained by moving exactly one box in the Young diagram of λ elsewhere. Note that if we apply the Schur functor to this, we get the well known formula for the Kronecker product $S^\lambda \otimes S^{(a,1)}$.

(ii) We calculate $\Delta(\lambda) \underline{\otimes} \Delta((p, 1^q))$ where p, q are positive integers with $d = p + q$. Straightforward imitation of earlier procedure would require us to involve Γ^ν where ν has several parts. Instead we use polynomial functors whose arguments are super-vector spaces (see [J13]). In this language

$\Gamma^p(V) \otimes \wedge^q(V) \simeq \Gamma_{(p,q)}^{d, k^+ \oplus k^-}(V)$. On the right hand side, the parametrization by $k^+ \oplus k^-$ and taking the (p, q) weight space amounts to requiring p letters from the argument V to commute and remaining q letters to anticommute. Similarly, using obvious terminology, $\Delta_+(\mu)(V) =$

$\Delta(\mu)(V_+) = \Delta(\mu)(V)$ whereas $\Delta_-(\mu)(V) = \Delta(\mu)(V_-) \simeq \nabla(\mu')(V)$, which is $\Delta(\mu')(V)$ for k a field of characteristic 0. Proceeding as before via a super-analogue of the relevant filtration, we have

$$\begin{aligned}
\Delta(\lambda) \underline{\otimes} (\Gamma^p \otimes \wedge^q) &\simeq \Gamma_{(p,q)}^{d, k^+ \oplus k^-} \\
&\simeq \bigoplus_{\mu \subset \lambda, |\mu|=p} \Delta_+(\mu) \otimes \Delta_-(\lambda/\mu) \\
&\simeq \bigoplus_{\mu \subset \lambda, |\mu|=p} \Delta_+(\mu) \otimes \Delta_+(\lambda'/\mu') \\
&\simeq \bigoplus_{\mu \subset \lambda, |\mu|=p} \Delta(\mu) \otimes \left(\bigoplus_{\nu \subset \lambda, |\nu|=q} c_{\mu', \nu}^{\lambda'} \Delta(\nu) \right) \\
&\simeq \bigoplus_{|\mu|=p, |\nu|=q, |\alpha|=d} c_{\mu', \nu}^{\lambda'} c_{\mu, \nu}^{\alpha} \Delta(\alpha)
\end{aligned}$$

Now again by Pieri's rule, $\Gamma^p \otimes \wedge^q \simeq \Delta((p, 1^q)) \oplus \Delta((p+1, 1^{q-1}))$. This allows one to calculate $\Delta(\lambda) \underline{\otimes} \Delta(p, 1^q)$ as the alternating sum of $\Delta(\lambda) \underline{\otimes} (\Gamma^{p+i} \otimes \wedge^{q-i})$ with $i = 0, \dots, q$.

References

- [AB85] Kaan Akin and David A Buchsbaum. Characteristic-free representation theory of the general linear group. *Advances in Mathematics*, 58(2):149–200, 11 1985.
- [AR15] C. Aquilano and R. Reischuk. The monoidal structure on strict polynomial functors. *arXiv:1503.05108*, 2015.
- [Bou67] A.K. Bousfield. *Homogeneous functors and their derived functors*. Brandies University, 1967.
- [Cha08] Chaalupnik. Koszul duality and extensions of exponential functors. *Adv. Math.*, 218(3):969–982, 2008.
- [CHN10] Frederick R. Cohen, David J. Hemmer, and Daniel K. Nakano. On the cohomology of Young modules for the symmetric group. *Adv. Math.*, 224(4):1419–1461, 2010.
- [FS97] Eric M. Friedlander and Andrei Suslin. Cohomology of finite group schemes over a field. *Invent. Math.*, 127(2):209–270, 1997.
- [Gre80] J.A. Green. *Polynomial representations of GL_n* . Number 830 in Lecture notes in Mathematics. Springer-Verlag, 1980.
- [J13] Axtell J. Spin polynomial functors and representations of schur superalgebras. *arxiv:1302.0042*, 2013.
- [Jam78] D.G James. *The representation theory of symmetric groups*. Number 682 in Lecture Notes in Mathematics. Springer-Verlag, 1978.

- [Kra13] Henning Krause. Koszul, Ringel and Serre duality for strict polynomial functors. *Compos. Math.*, 149(6):996–1018, 2013.
- [Kuh94] N.J. Kuhn. Generic representations of the finite general linear group and steenrod algebra i. *Amer. J. Math.*, 116(2):327–360, 1994.
- [Mac95] Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford mathematical monographs. Clarendon Press New York, 1995.
- [Mar09] Stuart Martin. *Schur algebras and representation theory*. Cambridge tracts in mathematics. Cambridge University Press, 2009.
- [Sag01] B.E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics. Springer, 2001.
- [Sta01] R.P Stanley. *Enumerative Combinatorics Vol II*. Cambridge Studies in Advanced Mathematics, 2001.
- [SY12] Ana Paula Santana and Ivan Yudin. Characteristic-free resolutions of Weyl and Specht modules. *Adv. Math.*, 229(4):2578–2601, 2012.
- [Tou12] Antoine Touzé. Troesch complexes and extensions of strict polynomial functors. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(1):53–99, 2012.
- [Tou13] Antoine Touzé. Ringel duality and derivatives of non-additive functors. *J. Pure Appl. Algebra*, 217(9):1642–1673, 2013.