

# Torus quotients of Richardson varieties in the Grassmannian

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October 25, 2018

## Abstract

We study the GIT quotient of the minimal Schubert variety in the Grassmannian  $G_{r,n}$  admitting semistable points for the action of maximal torus  $T$ , with respect to the  $T$ -linearized line bundle  $\mathcal{L}(n\omega_r)$ , and show that this is smooth when  $\gcd(r, n) = 1$ . When  $n = 7$  and  $r = 3$  we study the GIT quotients of all Richardson varieties in the minimal Schubert variety. This builds on previous work by Kumar [21], Kannan and Sardar [18], Kannan and Pattanayak [17], and recent work of Kannan et al. [16]. It is known that the GIT quotient of  $G_{2,n}$  is projectively normal. We give a different combinatorial proof.

**Key words**— GIT, Semistable points, Projective normality, Richardson varieties.

**Subject Classification**— 14M15

## 1 Introduction

Let  $G$  be a simply connected semi-simple algebraic group over  $\mathbb{C}$ . Let  $T$  be a maximal torus of  $G$ . Let  $B$  be a Borel subgroup of  $G$  containing  $T$ . We denote by  $B^-$  the Borel subgroup of  $G$  opposite to  $B$  determined by  $T$ . Let  $Q$  be a parabolic subgroup of  $G$  containing  $B$ . Then  $G/Q$  is a projective variety (see [12]). Let  $\mathcal{L}$  be a  $T$ -linearized ample line bundle on  $G/Q$ . A point  $p \in G/Q$  is said to be semistable with respect to the  $T$ -linearized line bundle  $\mathcal{L}$  if there is a  $T$ -invariant section of a positive power of  $\mathcal{L}$  which does not vanish at  $p$ . We denote

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by  $(G/Q)_T^{ss}(\mathcal{L})$  the set of all semistable points with respect to  $\mathcal{L}$ . A point in  $(G/Q)_T^{ss}(\mathcal{L})$  is said to be stable if its  $T$ -orbit is closed in  $(G/Q^{ss})_T(\mathcal{L})$  and its stabilizer in  $T$  is finite. Let  $(G/Q)_T^s(\mathcal{L})$  denote the set of all stable points with respect to  $\mathcal{L}$ . This paper is motivated by the question of understanding the GIT quotient of  $G/Q$  with respect to the  $T$ -linearized bundle  $\mathcal{L}$ .

When  $G$  is of Dynkin type A this problem has been well studied. There is a reasonable body of work when  $G$  is of type other than A. Although the results in this paper pertain only to the case when  $G$  is of type A we will nevertheless assume that  $G$  is of general type in this introduction to give a comprehensive survey of the results known. For that we will need to introduce some notation. We follow the notation from Lakshmibai and Raghavan [22].

Let  $X(T)$  denote the group of characters of  $T$ . In the root system  $R$  of  $(G, T)$  let  $R^+$  denote the set of positive roots with respect to  $B$ . Let  $S = \{\alpha_1, \dots, \alpha_l\} \subseteq R^+$  be the set of simple roots and let  $\{\omega_1, \dots, \omega_l\}$  be the fundamental weights. Let  $U$  (respectively,  $U^-$ ) be the unipotent radical of  $B$  (respectively,  $B^-$ ). For each  $\alpha \in R^+$ , let  $U_\alpha$  (respectively,  $U_{-\alpha}$ ) be the additive one-dimensional subgroup of  $U$  (respectively,  $U^-$ ) corresponding to the root  $\alpha$  (respectively,  $-\alpha$ ) normalized by  $T$ .

Let  $N_G(T)$  denote the normalizer of  $T$  in  $G$ . The Weyl group  $W$  of  $G$  is defined to be the quotient  $N_G(T)/T$ , and for every  $\alpha \in R$  there is a corresponding reflection  $s_\alpha \in W$ .  $W$  is generated by  $s_\alpha$ ,  $\alpha$  running over simple roots in  $S$ . This also defines a length function  $l$  and the Bruhat order on  $W$ .

For a subset  $I \subseteq S$  let  $W^I = \{w \in W | w(\alpha) > 0, \alpha \in I\}$  and  $W_I$  be the subgroup of  $W$  generated by  $s_\alpha$ ,  $\alpha \in I$ . Then every  $w \in W$  can be uniquely expressed as  $w = w^I w_I$ , with  $w^I \in W^I$  and  $w_I \in W_I$ . For  $w \in W$ , let  $n_w \in N_G(T)$  be a representative of  $w$ . We denote by  $P_I$  the parabolic subgroup of  $G$  generated by  $B$  and  $n_w$ ,  $w \in W_I$ . Then  $W_I$  is the Weyl group of the parabolic subgroup  $P_I$ . Sometimes we use the notation  $W_{P_I}$  instead of  $W_I$ . When  $I = S \setminus \{\alpha_r\}$ , we denote the corresponding maximal parabolic subgroup of  $G$  by  $P_{\hat{\alpha}_r}$ .

The quotient space  $G/P$  is a homogenous space for the left action of  $G$ . The  $T$ -fixed points in  $G/P$  are  $e_w = wP/P$  with  $w \in W^P$ . The  $B$ -orbit  $C_w$  of  $e_w$  is called a Bruhat cell, and it is an affine space of dimension  $l(w)$ . The closure of  $C_w$  in  $G/P$  is the Schubert variety  $X(w)$ . The opposite Bruhat cell  $C^w$  is the  $B^-$  orbit of  $e_w$ , and its closure, denoted by  $X^w$ , is the opposite Schubert variety. For a  $T$ -linearized line bundle  $\mathcal{L}$  on a Schubert variety in  $G/P$  we define the notion of semistable and stable points as before. We use the notation  $X(w)_T^{ss}(\mathcal{L})$  (respectively,  $X(w)_T^s(\mathcal{L})$ ) to denote the semistable (respectively, stable) points for the  $T$ -linearized line bundle  $\mathcal{L}$ .

Every character  $\lambda$  of  $P$  defines a  $G$ -linearized line bundle on  $G/P$ . We denote the line bundle by  $\mathcal{L}(\lambda)$ . Furthermore,  $\mathcal{L}(\lambda)$  is generated by global sections if and only if  $\lambda$  is a dominant weight (see [12, Part II, Proposition 2.6]).

When  $G = SL(n, \mathbb{C})$  and  $P = P_{\hat{\alpha}_r}$ ,  $G/P$  is the Grassmannian parametrizing  $r$ -dimensional subspaces of  $\mathbb{C}^n$ . We denote it by  $G_{r,n}$ . The Grassmannian  $G_{r,n}$  comes with the Plücker embedding  $G_{r,n} \hookrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n)$  sending each  $r$ -dimensional subspace to its  $r$ -th exterior wedge

product (see [5]). The pull back of  $\mathcal{O}(1)$  from the projective space to  $G_{r,n}$  is an ample generator of the Picard group of  $G_{r,n}$  and corresponds to the  $T$ -linearized line bundle  $\mathcal{L}(\omega_r)$ . Gelfand and MacPherson [7] considered the GIT quotient of the Grassmannian and showed that the GIT quotient of  $n$ -points in  $\mathbb{P}^{r-1}$  (spanning  $\mathbb{P}^{r-1}$ ) by the diagonal action of  $PGL(r, \mathbb{C})$  is isomorphic to the GIT quotient of  $G_{r,n}$  with respect to the  $T$ -linearized line bundle  $\mathcal{L}(n\omega_r)$ . They showed that the torus action gives rise to a moment map from  $G_{r,n}$  to  $\mathbb{R}^n$ , with the property that the image of each orbit is a convex polyhedron. This was extended by Gelfand et al. in [6]. In their paper the authors proposed three natural ways to stratify the Grassmannian - the first stratification is motivated by the equivalence of the torus quotient with the configuration of points in  $\mathbb{P}^{k-1}$ , the second is motivated by the moment map above, and the third is motivated by the geometry of intersections of Schubert cells in the Grassmannian. The authors show that no matter which definition is used to stratify the Grassmannian, the strata are the same.

Hausmann and Knutson [8] used the stratification from [6] to study the GIT quotient of  $G_{2,n}$  and related the resulting GIT quotient to the moduli space of polygons in  $\mathbb{R}^3$ .

Using the Hilbert-Mumford criterion, Skorobogatov [26] gave combinatorial conditions determining when a point in  $G_{r,n}$  is semistable with respect to the  $T$ -linearized bundle  $\mathcal{L}(\omega_r)$ . As a corollary he showed that when  $r$  and  $n$  are coprime semistability is the same as stability.

Independently, for a general  $G$ , Kannan (see [14] and [15]) gave a description of parabolic subgroups  $Q$  of  $G$  for which there exists an ample line bundle  $\mathcal{L}$  on  $G/Q$  such that  $(G/Q)_T^{ss}(\mathcal{L})$  is the same as  $(G/Q)_T^s(\mathcal{L})$ . In particular, in the case when  $G = SL(n, \mathbb{C})$  and  $Q = P_{\hat{\alpha}_r}$ , Kannan showed that  $(G_{r,n})_T^s(\mathcal{L}(\omega_r))$  is the same as  $(G_{r,n})_T^{ss}(\mathcal{L}(\omega_r))$  if and only if  $r$  and  $n$  are coprime.

In the type  $A$  case when  $G = SL(n, \mathbb{C})$  and  $Q$  is a parabolic subgroup, Howard [10] considered the problem of determining which line bundles on  $G/Q$  descend to ample line bundles of the GIT quotient of  $G/Q$  by  $T$ . For a line bundle which descends to an ample line bundle on the quotient, by the Gelfand-MacPherson correspondence, the smallest power of the descent bundle that is very ample would give an upper bound on the degree in which the ring of invariants of  $n$ -points spanning projective space  $\mathbb{P}^{r-1}$  is generated. Howard showed that when  $\mathcal{L}(\lambda)$  is a very ample line bundle on  $G/Q$  (so the character of  $T$  extends to  $Q$  and to no larger subgroup of  $G$ ) and  $H^0(G/Q, \mathcal{L}(\lambda))^T$  is non-zero, the line bundle descends to the quotient (see [10, Proposition 2.3, Theorem 2.3]). He extended these results to the case when the  $T$ -linearization of  $\mathcal{L}(\lambda)$  is twisted by  $\mu$ , a character of  $T$ . He proved that the line bundle  $\mathcal{L}(\lambda)$  twisted by  $\mu$  descends to the GIT quotient provided the  $\mu$ -weight space of  $H^0(G/Q, \mathcal{L}(\lambda))$  is non-zero, and this is so when  $\lambda - \mu$  is in the root lattice and  $\mu$  is in the convex hull of the Weyl orbit of  $\lambda$ . This was extended to other algebraic groups by Kumar [21, Theorem 3.10].

Kannan and Sardar [18] studied torus quotients of Schubert varieties in  $G_{r,n}$ . They showed that  $G_{r,n}$  has a unique minimal Schubert variety,  $X(w_{r,n})$ , admitting semistable

points with respect to the  $T$ -linearized bundle  $\mathcal{L}(\omega_r)$ , and they gave a combinatorial characterization of  $w_{r,n}$ .

Kannan and Pattanayak [17] extended the results of [18] to the cases when  $G$  is of Dynkin type  $B, C, D$  and when  $P$  is a maximal parabolic subgroup of  $G$ . Then  $G/P_{\hat{\alpha}_r}$  has an ample line bundle  $\mathcal{L}(\omega_r)$ . Kannan and Pattanayak gave a combinatorial description of all minimal Schubert varieties in  $G/B$  admitting semistable points with respect to  $\mathcal{L}(\lambda)$  for any dominant character  $\lambda$  of  $B$ .

Kannan et al. [16] extended the results in [18] to Richardson varieties in the Grassmannian  $G_{r,n}$ . Recall that a Richardson variety in  $G_{r,n}$  is the intersection of the Schubert variety  $X(w)$  in  $G_{r,n}$  with the opposite Schubert variety  $X^v$  in  $G_{r,n}$ . In [16] the authors gave a criterion for Richardson varieties in  $G_{r,n}$  to admit semistable points with respect to the  $T$ -linearized line bundle  $\mathcal{L}(\omega_r)$ .

## 1.1 Our results and Organization of the paper

For all the results in this paper we assume  $G$  is of Dynkin type A. In Section 3 we begin a study of the GIT quotient of  $G_{r,n}$  when  $r$  is bigger than 2 and  $\gcd(r, n) = 1$ . We study the GIT quotient of the minimal Schubert variety  $X(w_{r,n})$  having semistable points with respect to the  $T$ -linearized line bundle  $\mathcal{L}(n\omega_r)$ . We show that  $T \backslash\backslash X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r))$  is smooth. We show that  $w_{r,n} = cv_{r,n}$ , where  $c$  is a Coxeter element (i.e each simple reflection occurs exactly once in a reduced expression for  $c$ ) and  $l(w_{r,n}) = n - 1 + l(v_{r,n})$  with  $v_{r,n}$  being the (unique) maximal element  $v_{r,n} \in W^{S \setminus \alpha_r}$  such that  $v_{r,n}(n\omega_r) \geq 0$ . We show that the GIT quotient  $T \backslash\backslash (X_{w_{r,n}}^{v_{r,n}})_T^{ss}(\mathcal{L}(n\omega_r))$  is a point. We prove that the GIT quotient  $T \backslash\backslash (X_{w_{r,n}}^u)_T^{ss}(\mathcal{L}(n\omega_r))$  is  $\mathbb{P}^1$  precisely when one can write  $v_{r,n} = s_\alpha u$ , with  $l(v_{r,n}) = l(u) + 1$  and  $\alpha$  is a simple root. We determine all such simple roots and give a description of the descent line bundle to  $\mathbb{P}^1$ , in terms of the combinatorics of  $r, n$ .

In Section 4 we show that the polarized variety  $(T \backslash\backslash X(w_{3,7})_T^{ss}(\mathcal{L}(7\omega_3)), \tilde{\mathcal{L}}(7\omega_3))$  is projectively normal. In Section 5.2 we explicitly calculate the GIT quotients of the Richardson strata in  $X(w_{3,7})$  with respect to the  $T$ -linearized line bundle  $\mathcal{L}(7\omega_3)$ . We show that  $(T \backslash\backslash X(w_{3,7})_T^{ss}(\mathcal{L}(7\omega_3)), \tilde{\mathcal{L}}(7\omega_3))$  is a rational normal scroll. Finally, in Section 6 we prove that when  $n$  is odd, the polarized variety  $(T \backslash\backslash (G_{2,n})_T^{ss}(\mathcal{L}(n\omega_2)), \tilde{\mathcal{L}}(n\omega_2))$  is projectively normal, a result that is well known (see [10, 11]). However we believe that the combinatorics we develop to prove this result may be useful to extend this result to Grassmannians of higher ranks.<sup>1</sup>

**Acknowledgements:** S. Bakshi was supported by a research fellowship from the National Board of Higher Mathematics. All three authors were partially supported by a grant from the Infosys foundation. The third author was supported by a grant under the MATRICS scheme of the Department of Science and Technology. We would like to thank the anony-

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<sup>1</sup>In personal communication Pattanayak informs us that he and Arpita Nayek have a proof of projective normality of the GIT quotient of  $G_{2,n}$  and they have a counter example for Grassmannians of higher ranks.

mous referee for a very thorough reading of the manuscript and for pointing out errors in an earlier version of this paper. We also thank the referee for suggesting many changes that have vastly improved the readability of this paper.

## 2 Notations and Preliminaries

For the rest of this paper we will assume that  $G = SL(n, \mathbb{C})$  and  $P$  is a maximal subgroup of  $G$ . Keeping this in mind, we revisit the notation developed in the previous section for  $G$  of arbitrary type - we derive formulas for the various terms introduced and explicitly write down elements of the Weyl group and the action of the torus.

We take  $T$  to be the group of diagonal matrices in  $G$ ,  $B$  the subgroup of upper triangular matrices in  $G$  and  $B^-$  the subgroup of lower triangular matrices in  $G$ . The unipotent subgroup  $U$  is the subgroup of  $B$  with diagonal entries 1, and  $U^-$  is the unipotent subgroup of  $B^-$  with diagonal entries 1.  $S = \{\alpha_1, \dots, \alpha_{n-1}\}$  is the set of simple roots where  $\alpha_i = \epsilon_i - \epsilon_{i+1}$  (see [22, Chapter 3]). For each simple root  $\alpha_i$  we have a morphism  $\phi_i : SL(2, \mathbb{C}) \rightarrow G$ , with  $\phi_i$  sending  $M \in SL(2, \mathbb{C})$  to the  $n \times n$  matrix having  $M$  in rows and columns  $i, i+1$ , with the other diagonal entries being 1 and the remaining entries zero. We use the following notation:

$$x_i(m) = \phi_i \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad y_i(p) = \phi_i \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \quad \alpha_i^\vee(t) = \phi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \dot{s}_{\alpha_i} = \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (1)$$

Here  $m, p \in \mathbb{C}$  and  $t \in \mathbb{C}^*$ . It is easily checked that  $\dot{s}_{\alpha_i}$  is in  $N_G(T)$ . We use the notation  $s_{\alpha_i}$  for the coset  $\dot{s}_{\alpha_i}T$  in  $N_G(T)/T$ . The Weyl group  $W$  of  $G$  is generated by the  $s_{\alpha_i}$ . There is an isomorphism between  $W$  and the group of permutations of  $S_n$ , with  $s_{\alpha_i}$  mapping to the permutation swapping  $i$  with  $i+1$ . So  $W$  is identified with the permutation group  $S_n$  and for simplicity we also use the notation  $s_i$  for  $s_{\alpha_i}$ . We sometimes use the one line permutation notation  $(w(1), w(2), \dots, w(n))$  to denote  $w \in W$ .

Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $\mathbb{C}^n$ . Note that for  $r \in \{2, \dots, n\}$ ,  $P_{\hat{\alpha}_r} = \begin{bmatrix} * & * \\ 0_{n-r, r} & * \end{bmatrix}$  is the stabilizer of  $\langle e_1, e_2, \dots, e_r \rangle$  in  $G$ .  $G/P_{\hat{\alpha}_r}$  is the Grassmannian  $G_{r, n}$ , of  $r$ -dimensional subspaces of  $\mathbb{C}^n$  and this carries a transitive action of  $G$  making it a homogenous  $G$ -variety.  $W_{P_{\hat{\alpha}_r}}$  is the subgroup of  $W$  generated by simple reflections  $s_\alpha, \alpha \in S \setminus \{\alpha_r\}$ . Now  $W_{P_{\hat{\alpha}_r}} = S_r \times S_{n-r}$ , so the minimal length coset representatives of  $W/W_{P_{\hat{\alpha}_r}}$  can be identified with  $\{w \in W | w(1) < w(2) < \dots < w(r), w(r+1) < w(r+2) < \dots < w(n)\}$ .

Let  $I(r, n) = \{(i_1, i_2, \dots, i_r) | 1 \leq i_1 < i_2 < \dots < i_r \leq n\}$ . Then there is a natural identification of  $W^{S \setminus \{\alpha_r\}}$  with  $I(r, n)$  sending  $w \in W^{S \setminus \{\alpha_r\}}$  to  $(w(1), w(2), \dots, w(r))$ . For  $w$  in  $I(r, n)$ , let  $e_w = [e_{w(1)} \wedge e_{w(2)} \wedge \dots \wedge e_{w(r)}] \in \mathbb{P}(\bigwedge^r \mathbb{C}^n)$ . Then  $e_w$  is a  $T$ -fixed point of  $G_{r, n}$  and it is known that the  $e_w, w \in I(r, n)$  are precisely the  $T$ -fixed points of  $G_{r, n}$ . The  $B$ -orbit through  $e_w$  is the Schubert cell and its Zariski closure in  $G/P$  is the Schubert variety  $X(w)$ . Clearly the  $B$ -orbit of  $e_w$  is isomorphic to the  $U$ -orbit of  $e_w$ . The Bruhat order is the order on  $r$ -tuples in  $I(r, n)$  given by containment of Schubert varieties - in this order  $v \leq w$  if and only if

$v(i) \leq w(i)$  for  $1 \leq i \leq r$ . As mentioned in the previous section  $G_{r,n}$  comes with a natural line bundle  $\mathcal{L}(\omega_r)$  and a Plücker embedding  $p : G_{r,n} \hookrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n)^2$ . The coordinate function corresponding to  $\tau \in I(r, n)$  in this embedding is denoted by  $p_\tau$ . The collection of  $p_\tau$ 's are called the Plücker coordinates of the embedding. Each  $p_\tau$  is a weight vector for  $T$ .

Let  $\mathbb{C}[X(w)]$  be the homogeneous coordinate ring of  $X(w)$  for this projective embedding. From the main theorem of standard monomial theory for  $SL_n$  (see [22, Chapter 4]) we get  $H^0(X(w), \mathcal{L}(d\omega_r)) = \mathbb{C}[X(w)]_d$ , and this has a basis consisting of  $T$ -eigenvectors  $p_{\tau_1} p_{\tau_2} \cdots p_{\tau_d}$ , with  $\tau_1 \leq \tau_2 \leq \dots \leq \tau_d \leq w$ . Here the order  $\leq$  is the Bruhat order. The weight of  $p_{\tau_1} \cdots p_{\tau_d}$  is  $-\sum_i \tau_i(\omega_r)$ .

We associate with each standard monomial  $p_{\underline{\tau}} = p_{\tau_1} \cdots p_{\tau_d}$  a semistandard Young tableau  $T_{\underline{\tau}}$  of shape  $\underbrace{(d, d, \dots, d)}_{r \text{ times}}$  whose  $i$ -th column is filled with  $\tau_i = [\tau_i(1), \tau_i(2), \dots, \tau_i(r)]$  (see [25, Chapter 1]). It is clear that the rows of the semistandard Young tableau are weakly increasing and the columns are strictly increasing. Let  $a(i)$  denote the number of times integer  $i$  appears in the tableau. Then we have  $\text{diag}(t_1, \dots, t_n) \cdot p_{\tau_1} p_{\tau_2} \cdots p_{\tau_d} = \prod_i t_i^{a(i)} p_{\tau_1} \cdots p_{\tau_d}$ . Since  $t_1 \cdots t_n = 1$ , a standard monomial is a zero weight vector if and only if all  $a(i)$ 's appear the same number of times in the Young tableau.

First recall that given  $(b_1, \dots, b_r) \in I(r, n)$ , one reduced expression for the Weyl group element in  $W^{S \setminus \{\alpha_r\}}$  corresponding to this is  $(s_{b_1-1} \cdots s_1) \cdots (s_{b_r-1} \cdots s_r)$  where a bracket is assumed to be empty if  $b_i - 1$  is less than  $i$ .

We recall some lemmas and propositions which have appeared earlier. We state them nevertheless since they are required in the rest of the paper. Some of these are folklore.

The following lemma appears in [21], [14].

**Lemma 2.1.** *Let  $r$  and  $n$  be coprime. Let  $v \neq 0$  be a zero weight vector in  $H^0(X(w), \mathcal{L}(\omega_r)^{\otimes d})$ . Then  $n$  divides  $d$ .*

*Proof.* Since 0 is a weight,  $d\omega_r$  is in the root lattice. So  $n$  divides  $d$ . □

Recall from [18], that there is a unique minimal Schubert Variety  $X(w_{r,n})$  in  $G_{r,n}$  admitting semistable points with respect to the line bundle  $\mathcal{L}(n\omega_r)$ . For completeness we explicitly calculate  $w_{r,n}$ .

**Proposition 2.2.** *Let  $r$  and  $n$  be coprime. Then  $w_{r,n} = (a_1, a_2, \dots, a_r)$  where  $a_i$  is the smallest integer such that  $a_i r \geq in$ .*

*Proof.* Clearly  $w_{r,n} > id$  since  $X(id)$  is a point. Let  $\alpha$  be a simple root with  $s_\alpha w_{r,n} \leq w_{r,n}$ . Note,  $s_\alpha w_{r,n} \in W^{S \setminus \{\alpha_r\}}$ . We have a surjection  $H^0(X(w_{r,n}), \mathcal{L}(n\omega_r)) \rightarrow H^0(X(s_\alpha w_{r,n}), \mathcal{L}(n\omega_r))$ . Let  $K$  denote its kernel. So we have a short exact sequence

$$0 \rightarrow K \rightarrow H^0(X(w_{r,n}), \mathcal{L}(n\omega_r)) \rightarrow H^0(X(s_\alpha w_{r,n}), \mathcal{L}(n\omega_r)) \rightarrow 0.$$

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<sup>2</sup>This notation, valid for type A, is consistent with the notation set up in the introduction for all types.

From the minimality of  $w_{r,n}$  we get  $K^T \rightarrow H^0(X(w_{r,n}), \mathcal{L}(n\omega_r))^T$  an isomorphism. Now if we choose a standard monomial  $p_{\underline{\tau}} = p_{\tau_1} \cdots p_{\tau_r}$  in  $H^0(X(w_{r,n}), \mathcal{L}(n\omega_r))^T$  then we have  $\tau_r = w_{r,n}$ , since the elements in the kernel are precisely those with a term  $p_{w_{r,n}}$ .

To construct such a standard monomial, we need a filling of the associated tableau  $T_{\underline{\tau}}$  with  $rn$  boxes such that each  $i$ ,  $1 \leq i \leq n$  appears exactly  $r$ -times and the last column is as small as possible in the Bruhat order. Clearly, the filling which results in the smallest element in the Bruhat order appearing as the last column is the one in which the tableau is filled from left to right and top to bottom with numbers  $1, 2, \dots, n$ , in order, with each appearing exactly  $r$  times - so the first entry of the last column is the least integer  $a_1$  such that  $a_1 r \geq n$  and, in general, the  $i$ -th entry in the last column is the smallest integer  $a_i$  such that  $a_i r \geq in$ , completing the proof.  $\square$

The tableau constructed in the proof of Proposition 2.2 will be used repeatedly in the paper. We denote it by  $\Gamma_{r,n}$ . The figure below gives  $\Gamma_{3,8}$ .

$$\Gamma_{3,8} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 \\ \hline 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 \\ \hline \end{array}$$

### 3 GIT quotients of Richardson varieties in $X(w_{r,n})$

The results in this section pertain to GIT quotients of Richardson varieties in  $G/P_{\hat{\alpha}_r}$  with respect to the  $T$ -linearized line bundle  $\mathcal{L}(n\omega_r)$ .

#### 3.1 GIT quotients of Richardson varieties

**Theorem 3.1.** *Let  $r$  and  $n$  be coprime. Then the GIT quotient  $T \backslash\backslash X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r))$  is smooth.*

*Proof.*  $X(w_{r,n})$  is the minimal Schubert variety admitting semistable points with respect to  $\mathcal{L}(n\omega_r)$ . So  $X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r)) \cap BwP_{\hat{\alpha}_r}/P_{\hat{\alpha}_r} = \phi$  for all  $w < w_{r,n}$ . From the Bruhat decomposition it follows that  $X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r)) \subseteq Bw_{r,n}P_{\hat{\alpha}_r}/P_{\hat{\alpha}_r}$ . Thus,  $X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r))$  is a smooth open subset of  $X(w_{r,n})$ . Since  $r$  and  $n$  are coprime we have  $X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r)) = X(w_{r,n})_T^s(\mathcal{L}(n\omega_r))$  (see [13]). Let  $G_{ad} = G/Z(G)$  be the adjoint group of  $G$ . Let  $\pi : G \rightarrow G_{ad}$  be the natural homomorphism and  $T_{ad} = \pi(T)$ . Note that  $\mathcal{L}(n\omega_r)$  is also  $T_{ad}$ -linearized. Therefore,  $X(w_{r,n})_{T_{ad}}^{ss}(\mathcal{L}(n\omega_r)) = X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r)) = X(w_{r,n})_T^s(\mathcal{L}(n\omega_r)) = X(w_{r,n})_{T_{ad}}^s(\mathcal{L}(n\omega_r))$ . Hence for any point  $x \in X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r))$  the orbit  $T_{ad} \cdot x$  is closed in  $X(w_{r,n})_{T_{ad}}^{ss}(\mathcal{L}(n\omega_r))$  and the stabiliser of  $x$  is finite. By [13, Lemma 3.2] and the proof of example 3.3 in that paper, the stabiliser of every point of  $X(w_{r,n})_{T_{ad}}^{ss}(\mathcal{L}(n\omega_r))$  in  $T_{ad}$  is

trivial. Therefore the GIT quotient  $T \backslash\backslash X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r))$  is a geometric quotient. Since  $X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r))$  is smooth,  $T \backslash\backslash X(w_{r,n})_T^{ss}(\mathcal{L}(n\omega_r))$  is also smooth.  $\square$

Recall that a Richardson variety  $X_w^v$  in  $G_{r,n}$  is the intersection of the Schubert variety  $X(w)$  in  $G_{r,n}$  with the opposite Schubert variety  $X^v$  in  $G_{r,n}$ .

In [16, Proposition 3.1] the authors give a characterisation of the smallest Richardson variety in  $G_{r,n}$  admitting semistable points. From the proof of Proposition 2.2 we obtain an explicit characterization.

**Proposition 3.2.** *Let  $r$  and  $n$  be coprime. Let  $v_{r,n}$  be such that  $X_{w_{r,n}}^{v_{r,n}}$  is the smallest Richardson variety in  $X(w_{r,n})$  admitting semistable points. Then  $v_{r,n} = [1, a_1, \dots, a_{r-1}]$  with the  $a_i$  defined as the smallest integer satisfying  $a_i r \geq in$  (as in Proposition 2.2).*

*Proof.* Let  $v_{r,n} = [b_1, \dots, b_r]$ . Since  $X_{w_{r,n}}^{v_{r,n}}$  has a semistable point,  $H^0(X_{w_{r,n}}^{v_{r,n}}, \mathcal{L}(n\omega_r))^T$  is non-zero. Now  $H^0(X_{w_{r,n}}^{v_{r,n}}, \mathcal{L}(n\omega_r))$  has a standard monomial basis  $p_{\tau_1} \dots p_{\tau_n}$  with  $\tau_1 \leq \tau_2 \dots \leq \tau_n$  (see [2]). We identify this basis with semistandard Young tableau having columns  $\tau_1, \tau_2, \dots, \tau_n$  as before. It follows from this identification that there is a semistandard Young tableau with  $r$  rows and  $n$ -columns in which each integer  $1 \leq k \leq n$  appears exactly  $r$  times. From Proposition 2.2 and [2, Proposition 6] we have  $\tau_n = w_{r,n}$  and  $v_{r,n} \leq \tau_1$ . Since every semistandard Young tableau has each integer in  $\{1, \dots, n\}$  appearing  $r$  times and the first entry of  $\tau_1$  is always 1,  $b_1$  must be 1. Since  $r, n$  are coprime, from the definition of  $a_1$  it is immediate that all  $a_1$ 's cannot be in the first row. For the same reason the  $a_i$ 's cannot all appear in the first  $i$  rows. So  $a_i$  must appear in a row  $j$  where  $j > i$ . Hence  $b_i \leq a_{i-1}$ . Note that the first column of the Young tableau  $\Gamma_{r,n}$  from Proposition 2.2 is  $v = [1, a_1, \dots, a_{r-1}]$ . From [2, Proposition 6], the  $T$ -invariant  $\Gamma_{r,n}$  is non-zero on  $X_{w_{r,n}}^v$ . Hence  $v_{r,n} = v = [1, a_1, \dots, a_{r-1}]$ .  $\square$

Consider the Weyl group element  $c_{r,n} = w_{r,n} v_{r,n}^{-1}$ .

**Claim 3.3.**  $c_{r,n}$  is a Coxeter element.

*Proof.* We have a reduced expression  $w_{r,n} = (s_{a_1-1} \dots s_1)(s_{a_2-1} \dots s_2) \dots (s_{a_r-1} \dots s_r)$  and  $v_{r,n} = (s_{a_1-1} \dots s_2)(s_{a_2-1} \dots s_3) \dots (s_{a_{r-1}-1} \dots s_r)$ . Then

$$w_{r,n} v_{r,n}^{-1} = (s_{a_1-1} \dots s_1)(s_{a_2-1} \dots s_{a_1})(s_{a_3-1} \dots s_{a_2}) \dots (s_{a_r-1} \dots s_{a_{r-1}}).$$

This is a Coxeter element.  $\square$

We now consider the GIT quotients of Richardson varieties in  $X_{w_{r,n}}$ .

**Theorem 3.4.**  $T \backslash\backslash (X_{w_{r,n}}^{v_{r,n}})_T^{ss}(\mathcal{L}(n\omega_r))$  is a point.



*Proof.* Since  $\dim X_{w_{r,n}}^{v_{r,n}} = l(w_{r,n}) - l(v_{r,n}) = l(c_{r,n}) = n - 1 = \dim T$  and  $(X_{w_{r,n}}^{v_{r,n}})_{T}^{ss}(\mathcal{L}(n\omega_r)) = (X_{w_{r,n}}^{v_{r,n}})_{T}^s(\mathcal{L}(n\omega_r))$ , so the dimension of the GIT quotient is 0. Since  $T \backslash (X_{w_{r,n}}^{v_{r,n}})_{T}^{ss}(\mathcal{L}(n\omega_r))$  is irreducible,  $T \backslash (X_{w_{r,n}}^{v_{r,n}})_{T}^{ss}(\mathcal{L}(n\omega_r))$  is a point. Alternatively, there is a unique standard monomial  $p_{\tau_1} p_{\tau_2} \dots p_{\tau_n}$  of weight zero with  $\tau_1 = [1, a_1, a_2, \dots, a_{r-1}]$  and  $\tau_n = [a_1, a_2, \dots, a_r]$  (the corresponding Young tableau being  $\Gamma_{r,n}$ ).  $\square$

**Theorem 3.5.** *Let  $v \in W^{S \setminus \{\alpha_r\}}$  be such that  $v < v_{r,n}$ . Then  $T \backslash (X_{w_{r,n}}^v)_{T}^{ss}(\mathcal{L}(n\omega_r))$  is isomorphic to  $\mathbb{P}^1$  if and only if  $v = s_{\alpha} v_{r,n}$  where  $s_{\alpha} = (a_i - 1, a_i)$  for some  $i = 1, 2, \dots, r - 1$ . The descent of  $\mathcal{L}(n\omega_r)$  to  $T \backslash (X_{w_{r,n}}^v)_{T}^{ss}(\mathcal{L}(n\omega_r))$  is  $\mathcal{O}_{\mathbb{P}^1}(n_i)$  where  $n_i$  is the number of times  $a_i - 1$  appears in the  $i$ -th row of the tableau  $\Gamma_{r,n}$ .*

*Proof.* We start with the only if part. Since  $X_{w_{r,n}}^v$  is normal,  $(X_{w_{r,n}}^v)_{T}^{ss}(\mathcal{L}(n\omega_r))$  is normal and hence  $T \backslash (X_{w_{r,n}}^v)_{T}^{ss}(\mathcal{L}(n\omega_r))$  is normal. Since  $\dim(T \backslash (X_{w_{r,n}}^v)_{T}^{ss}(\mathcal{L}(n\omega_r))) = 1$ , the GIT quotient  $T \backslash (X_{w_{r,n}}^v)_{T}^{ss}(\mathcal{L}(n\omega_r))$  is a smooth, rational projective curve. Hence  $T \backslash (X_{w_{r,n}}^v)_{T}^{ss}(\mathcal{L}(n\omega_r))$  is isomorphic to  $\mathbb{P}^1$ .

If  $T \backslash (X_{w_{r,n}}^v)_{T}^{ss}(\mathcal{L}(n\omega_r))$  is isomorphic to  $\mathbb{P}^1$  we get  $l(v) = l(v_{r,n}) - 1$ . Also  $v < v_{r,n}$  and  $v \in W^{S \setminus \{\alpha_r\}}$ . So  $v = (s_{b_i} \dots s_i) \dots (s_{b_r} \dots s_r)$  for some  $i$ ,  $1 \leq i \leq r$ , and for some  $1 \leq b_i < b_{i+1} \dots < b_r \leq n - 1$  (see the discussion preceding Lemma 2.1). Since  $v_{r,n} = (s_{a_1-1} \dots s_2)(s_{a_2-1} \dots s_3) \dots (s_{a_{r-1}-1} \dots s_r)$ ,  $v = s_{\alpha} v_{r,n}$  only when  $s_{\alpha} = (a_i - 1, a_i)$ ,  $1 \leq i \leq r - 1$ .

We start with a zero weight standard monomial basis for  $H^0(X_{w_{r,n}}^v, \mathcal{L}(n\omega_r))$ . Let  $v = s_{\alpha} v_{r,n}$  with  $s_{\alpha} = (a_i - 1, a_i)$  for some fixed  $i$ . We have  $v_{r,n} = (1, a_1, \dots, a_i, \dots, a_{r-1})$  and  $v = (1, a_1, \dots, a_i - 1, \dots, a_{r-1})$ . The  $i + 1$ -st entry of  $v_{r,n}$  is  $a_i$  and that of  $v$  is  $a_i - 1$  and the rest of the entries are equal. We need to count the number of semistandard tableau of shape  $n, n, \dots, n$  ( $r$  rows) with first column  $v$ . Because the tableau is semistandard, the positions of all integers other than  $a_i - 1$  and  $a_i$  are fixed. So the number of such tableaux depends only on the number of  $a_i - 1$  in the  $i$ -th row.  $a_i - 1$  appears  $n_i$  times in the  $i$ -th row of  $\Gamma_{r,n}$ . It is easy to see that for every  $j$  in  $\{0, \dots, n_i\}$  there is a semistandard tableau with  $a_i - 1$  appearing  $j$  times and  $a_i$  appearing  $n_i - j$  times in row  $i$ . So we have  $n_i + 1$  linearly independent sections of the descent line bundle on the GIT quotient. This completes the proof.  $\square$

## 4 Projective normality of the GIT quotient of $X(w_{3,7})$

In this section we will work with  $G = SL(7, \mathbb{C})$ . We use the same notation as before. We study the GIT quotient of the Schubert variety  $X(w_{3,7})$  with respect to  $T$ -linearized line bundle  $\mathcal{L}(7\omega_3)$ . From [21, Theorem 3.10] we know that this line bundle descends to the line bundle  $\tilde{\mathcal{L}}(7\omega_3)$  on the GIT quotient  $T \backslash X(w_{3,7})_{T}^{ss}(\mathcal{L}(7\omega_3))$ .

**Theorem 4.1.** *The polarized variety  $(T \backslash\backslash X(w_{3,7})_T^{ss}(\mathcal{L}(7\omega_3)), \tilde{\mathcal{L}}(7\omega_3))$  is projectively normal.*

**Remark 4.2.** *Let  $S(m) = H^0(X(w_{3,7}), \mathcal{L}(7\omega_3)^{\otimes m})$  be the global sections of the line bundle  $\mathcal{L}(7\omega_3)$  on  $X(w_{3,7})$  and let  $R(m) = H^0(X(w_{3,7}), \mathcal{L}(7\omega_3)^{\otimes m})^T$  denote the invariant subspace with respect to action of  $T$ . The GIT quotient is precisely  $\text{Proj}(\oplus_m R(m))$  (see [4, Proposition 8.1]). Since the polarized variety  $(X(w_{3,7}), \mathcal{L}(7\omega_3))$  is projectively normal, we have a surjective map  $S(1)^{\otimes m} \rightarrow S(m)$  (see [22]) and an induced map  $\phi : R(1)^{\otimes m} \rightarrow R(m)$ . Now the GIT quotient is smooth and it is normal. Therefore to show projective normality of the GIT quotient all we need to show is that  $\phi$  is surjective.*

From Lemma 2.2 we get  $w_{3,7} = [3, 5, 7]$ . As before we identify the standard monomial basis of  $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$  with semistandard Young tableaux. These tableaux have 3 rows and  $7m$  columns with each integer from  $\{1, \dots, 7\}$  appearing exactly  $3m$  times - furthermore the last column is  $[3, 5, 7]$ .

To aid in the proof of projective normality we list the semistandard Young tableau basis of  $R(1)$  and we also write down a semistandard tableau of shape  $[14, 14, 14]$  from  $R(2)$  which will play a role in the proof. Henceforth, we will use the notation  $y_i$  for both the tableau  $y_i$  it defines and also the standard monomial associated it to. Set

$$\begin{aligned}
y_1 &= \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 3 & 3 & 4 & 4 & 4 & 5 & 5 \\ \hline 5 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, & y_2 &= \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 3 & 3 & 4 & 4 & 5 & 5 & 5 \\ \hline 4 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, & y_3 &= \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 & 4 & 5 & 5 \\ \hline 5 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, \\
y_4 &= \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 & 5 & 5 & 5 \\ \hline 4 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, & y_5 &= \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 & 4 & 5 & 5 \\ \hline 5 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, & y_6 &= \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 & 5 & 5 & 5 \\ \hline 4 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, \\
y_7 &= \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & 5 & 5 & 5 \\ \hline 3 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, & z &= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \\ \hline 3 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 \\ \hline \end{array}.
\end{aligned}$$

We first make some simple observations.

**Observation 4.3.** *Every semistandard tableau basis element of  $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$  begins with one of the following columns -  $[1, 2, 3]$ ,  $[1, 2, 4]$ ,  $[1, 2, 5]$ ,  $[1, 3, 4]$ ,  $[1, 3, 5]$ , and ends with the column  $[3, 5, 7]$ .*

*Proof.* We already noted above that the last column of every semistandard tableau basis element of  $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$  is  $[3, 5, 7]$ .

Clearly, semistandardness forces that in the first row the leftmost  $3m$  entries are filled with 1, and that in the last row the rightmost  $6m$  entries are filled with  $3m$  6's followed by  $3m$  7's. So clearly the last entry of the first column cannot be 6 or 7 otherwise we will have

more 6's or 7's than permitted. The second entry of the first column cannot be 5, otherwise the entire second row will have only 5's, a contradiction to the number of 5's present. The second entry of the first column cannot be 4 for a similar reason - in that case the second row will only have 4's and 5's forcing at least one of them to occur more than  $3m$  times, a contradiction. This completes the proof.  $\square$

**Observation 4.4.** *No semistandard tableau basis element of  $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$  has the following columns:  $[1, 2, 7]$ ,  $[1, 3, 7]$ ,  $[1, 4, 7]$ ,  $[1, 5, 6]$ ,  $[1, 5, 7]$ ,  $[2, 3, 4]$ ,  $[2, 3, 5]$ ,  $[2, 3, 6]$ ,  $[2, 3, 7]$ ,  $[2, 4, 5]$ ,  $[2, 5, 6]$ ,  $[3, 4, 6]$ ,  $[3, 5, 6]$ . The only columns containing a 6 are columns  $[1, 2, 6]$ ,  $[1, 3, 6]$ ,  $[1, 4, 6]$  and  $[2, 4, 6]$ . There are exactly  $m$  columns with  $[2, 4, 6]$  and at least  $m$  columns with  $[1, 4, 6]$ . The only columns containing a 7 are  $[2, 4, 7]$ ,  $[2, 5, 7]$ ,  $[3, 4, 7]$  and  $[3, 5, 7]$  and there are at least  $2m$  occurrences of columns  $[2, 5, 7]$  and  $[3, 5, 7]$ .*

*Proof.* If there is a column with  $[1, x, 7]$ ,  $x$  among  $2, 3, 4, 5, 6$ , standardness forces that the entries in the first row to the left of this column are all 1's and the entries in the third row to its right are all 7. Then no matter where this column appears either the number of 1's or the number of 7's is incorrect.

It follows from the previous paragraph that all the 6's must be in the bottom row. If there is a column with  $[1, 5, 6]$ , then standardness forces the subsequent columns to all have a 5 in the second row and the columns preceding it to have a 1 in the topmost row. Then no matter where this column appears either the number of 1's or the number of 5's is not  $3m$ .

If  $[2, 3, 4]$  occurs it is necessarily in column  $3m + 1$  appearing immediately after the occurrence of all the columns containing 1 because it is lexicographically least among columns beginning with 2. But the entry in the bottom row position in column  $3m + 1$  cannot be a 4 since that position is occupied by 6.

The same argument shows that  $[2, 3, 5]$ ,  $[2, 4, 5]$  cannot occur. If  $[2, 3, 6]$  or  $[2, 3, 7]$  is present, the first row to the right of this column and the second row to the left of this column contain only 2,3's yielding a total of  $7m$  entries with 2 and 3, a contradiction.

If  $[2, 5, 6]$  is present all columns to the right of this column will have a 5 in the second row by standardness. But then all the  $3m$  columns containing 7 will be of the type  $[x, 5, 7]$ , for some  $x$ . But then the number of 5's is at least  $3m + 1$ , a contradiction.

If the column  $[3, 4, 6]$  is present then the top row to its right is filled with 3's. So every column containing 7 in the bottom has 3 as its topmost element. So the number of 3's is at least  $3m + 1$ , a contradiction.

Now suppose the column  $[3, 5, 6]$  is present. If it is in the left half of the tableau, standardness will force the number of 5's to be more than what is allowed. If it is in the right half of the tableau then all entries in the second row to the right of this column are filled with 5's. So all the columns containing 7 in fact contain both 5 and 7. Again, the number of 5's is more than  $3m + 1$ , a contradiction.

The above argument shows that the column appearing immediately after all the columns containing a 1 is column  $[2, 4, 6]$ . It appears before the  $3m$  columns containing a 7. Since

the tableau has no lexicographically larger column containing a 6, this column repeats till the appearance of a 7. So it occurs exactly  $m$  times.

Now the remaining  $2m$  columns containing a 6 in the last row occur to the left of column number  $3m + 1$  which has a  $[2, 4, 6]$ . Suppose there are less than  $m$  columns with  $[1, 4, 6]$  in the tableau. Since  $[1, 4, 6]$  appears to left of the column numbered  $3m + 1$ , all the entries in the second row to the left of first column labeled  $[1, 4, 6]$  must have 2 or 3. So there are at least  $2m + 1$  2's and 3's in the second row. Now there are at least  $4m$  locations in the first row to the right of last 1 which have only 2 or 3. So the total number of 2's and 3s is at least  $6m + 1$ , a contradiction. So we conclude that at least  $m$  rows to the left of column numbered  $3m + 1$  contain  $[1, 4, 6]$ .

We cannot have a 5 in the first row. Since we can have a 5 in the third row only in positions  $\{1, 2, \dots, m\}$ , and the only columns having a 5 in the second row are  $[2, 5, 7]$  and  $[3, 5, 7]$  it follows that we need at least  $3m - m$  columns with  $[2, 5, 7]$  and  $[3, 5, 7]$ .  $\square$

**Lemma 4.5.** *Let  $m \geq 2$ . Every semistandard basis element of  $R(m)$  is a product of a  $y_i$  and an element of  $R(m - 1)$ , or is a product of  $z$  and an element of  $R(m - 2)$ .*

*Proof.* Let  $f$  be a semistandard basis element of  $R(m)$ . The proof follows a case by case analysis.

- a The first column of  $f$  is  $[1, 2, 3]$ . By Observation 4.4 above we have at least  $m$  columns with  $[1, 4, 6]$  and exactly  $m$  columns labeled  $[2, 4, 6]$ . Furthermore we can have at most  $m - 1$  5's in the last row of  $f$ . So we have at least  $2m + 1$  columns in  $f$  with  $[2, 5, 7]$  and  $[3, 5, 7]$ , since these are the only columns containing 5 in the second row. The last column of  $f$  is a  $[3, 5, 7]$ . If the remaining  $2m$  columns were all  $[2, 5, 7]$ , using Observation 4.4, the total number of 2's is at least  $m + 1 + 2m$ , a contradiction. (the  $m$  2's from columns with  $[2, 4, 6]$ , and one from the first column having  $[1, 2, 3]$ ). It follows that there are at least two columns with  $[3, 5, 7]$ .
  1. Suppose  $f$  has at least one column with  $[2, 5, 7]$ . Then we have one  $[1, 2, 3]$ , at least two  $[1, 4, 6]$ 's and one  $[2, 4, 6]$ , one  $[2, 5, 7]$  and two  $[3, 5, 7]$ 's. So the tableau  $y_7$  appears as a subtableau. The complement of this subtableau in  $f$  is an element of  $R(m - 1)$ . So  $f$  is a product of  $y_7$  and an element from  $R(m - 1)$  as required.
  2.  $f$  has no  $[2, 5, 7]$ . So we have at least  $2m + 1$  columns in  $f$  having  $[3, 5, 7]$ 's. Now the remaining 7's can be made up from  $[3, 5, 7]$ 's or  $[2, 4, 7]$ 's or  $[3, 4, 7]$ 's. These cannot all come from  $[3, 5, 7]$  and  $[3, 4, 7]$  since the number of 3's in that case would be more than  $3m + 1$ . So there is at least one  $[2, 4, 7]$ . Note that there are also at most  $m - 1$   $[2, 4, 7]$ 's,  $[3, 4, 7]$ 's, and additional  $[3, 5, 7]$ 's in from column numbers  $4m + 1$  to  $5m - 1$ . Now the number of 2's in row 1 is at most  $m + m - 1$  (from the columns with  $[2, 4, 6]$ , and at most  $m - 1$  columns with  $[2, 4, 7]$ ). So we need at least  $m + 1$  2's in the second row. In this case then the second row of columns 1 to column  $m + 1$  contains only 2. In particular  $[1, 2, 6]$  is present in  $f$ . Since there are

at most  $3m - 1$  5's in the second row of  $f$  (since we know there is a  $[2, 4, 7]$ ), there is at least one 5 in the bottom row of  $f$  in position  $\{1, 2, \dots, m\}$ , forcing a  $[1, 2, 5]$  in  $f$ .

Now the total number of 4's and 5's is  $6m$ . The total number of 4's and 5's in the last row is at most  $m - 1$ . We have  $m$  4's from the  $[2, 4, 6]$ . The total number of 4's and 5's from the columns containing  $[2, 4, 7]$ ,  $[3, 4, 7]$  and  $[3, 5, 7]$  is at most  $3m$ . All of these can account for a total of  $5m - 1$  4's and 5's coming from these columns. Since we can have no more 5's in the second row, the deficit  $m + 1$  needed must be made from 4's in the second row, in fact occurring in columns numbered  $m + 2$  to  $3m$ . So we have at least 3 columns in  $f$  with  $[1, 4, 6]$ 's.

Taking stock, in  $f$  we have one  $[1, 2, 3]$ , a  $[1, 2, 5]$ , a  $[1, 2, 6]$ , 3 columns with  $[1, 4, 6]$ , 2 columns with  $[2, 4, 6]$ , one  $[2, 4, 7]$  and at least 5  $[3, 5, 7]$ 's. So we see that the tableau indexing the basis vector  $z$  is a subtableau of  $f$ , and the complement of this subtableau in  $f$  is an element of  $R(m - 2)$ .  $f$  is a product of  $z$  and an element from  $R(m - 2)$ .

- b The first column of  $f$  is  $[1, 2, 4]$ . In this case there are at most  $m - 1$  5's in the last row of  $f$  and so there should be at least  $2m + 1$  columns in  $f$  with  $[2, 5, 7]$  and  $[3, 5, 7]$ .

Notice that the  $3m$  6's cannot all come from columns with  $[1, 4, 6]$  and  $[2, 4, 6]$  alone. If that were the case we will have  $3m$  4's from these columns, and an additional 4 from the first column, a contradiction. So at least one of the columns in  $f$  with a 6 has to be  $[1, 2, 6]$  or  $[1, 3, 6]$ .

1. Suppose a  $[1, 2, 6]$  is present in  $f$ . Then it has to be in column  $m + 1$  of  $f$ . Then we have 2's in the second row of  $f$  in columns 1 to columns  $m + 1$  by semistandardness. From Observation 4.4 we have  $m$  2's from the columns with  $[2, 4, 6]$ , so we can have at most  $m - 1$  columns with  $[2, 5, 7]$ . This means there are at least  $2m + 1 - (m - 1) = m + 2$  columns with  $[3, 5, 7]$ , so we have at least 4 columns with  $[3, 5, 7]$ . But this means we have a  $[1, 2, 4]$ , a  $[1, 2, 6]$ , a  $[1, 4, 6]$ , a  $[2, 4, 6]$ , and three  $[3, 5, 7]$ , i.e. the tableau indexing the basis element  $y_6$  is a subtableau in this case.
2. Now suppose we do not have a  $[1, 2, 6]$  in  $f$  but have a  $[1, 3, 6]$ . We claim a  $[2, 5, 7]$  must appear. Notice that we can have at most  $m - 1$   $[2, 4, 7]$ 's since we already have  $2m + 1$  7's. Now there are at most  $m$  2's in the second row of  $f$ . But this means we have at most  $m + m - 1 + m < 3m$  2's in  $f$ , a contradiction. So we may assume we have at least one  $[2, 5, 7]$  in  $f$ . We claim that we have at least 2  $[3, 5, 7]$ 's in  $f$ , for otherwise we have  $2m$   $[2, 5, 7]$ 's. But we have more 2's than allowed since we have  $m$  2's from the  $[2, 4, 6]$  and a 2 also from the  $[1, 2, 4]$ . So we conclude we have a  $[1, 2, 4]$ ,  $[1, 3, 6]$ ,  $[1, 4, 6]$ ,  $[2, 4, 6]$ ,  $[2, 5, 7]$ ,  $[3, 5, 7]$ ,  $[3, 5, 7]$ . So  $y_4$  is a subtableau and we are done in this case.

- c If the first column in  $f$  is  $[1, 2, 5]$ . If there are no  $[1, 2, 6]$  or  $[1, 3, 6]$  in  $f$  then column  $m + 1$  must be  $[1, 4, 6]$  and the first  $m$  elements in the third row must be all 5's. But

then the second and third row together have more than  $6m$  4's and 5's, a contradiction. So either  $[1, 2, 6]$  or  $[1, 3, 6]$ , or both, are present in  $f$ .

1. Suppose first that  $f$  has a  $[1, 2, 6]$ . Then the second row of  $f$  has at least  $m + 1$  2's, and since we already have  $m$  2's from the  $[2, 4, 6]$ 's we can have at most  $m - 1$   $[2, 5, 7]$ 's. This forces at least  $m + 1$  columns of  $f$  to be  $[3, 5, 7]$ 's.

If  $f$  has a column with  $[3, 4, 7]$  we see the tableau indexing  $y_5$  is present as a subtableau of  $f$ . If  $f$  has no  $[3, 4, 7]$  - we count the number of 4's - we have  $m$  from columns  $[2, 4, 6]$ . We can have at most  $2m - 1$  columns with  $[1, 4, 6]$  since 2's occupies positions 1 up to  $m + 1$  in the second row. To make up the requisite 4's we need to have at least one  $[2, 4, 7]$ . But then the number of 3's in the first row of the tableau from  $[3, 5, 7]$  is at most  $3m - 1$ , so to make up the requisite 3's, there must be a  $[1, 3, 6]$ . In which case we see that the tableau indexing  $y_3$  is present as a subtableau of  $f$ .

2. Suppose we only have a  $[1, 3, 6]$  in  $f$  and no  $[1, 2, 6]$ . Now the total number of columns in  $f$  with  $[1, 2, 5]$  and  $[1, 3, 5]$  is  $m$ . We have exactly  $2m$   $[2, 5, 7]$ 's and  $[3, 5, 7]$ 's put together. If  $f$  has no  $[2, 4, 7]$ , the remaining  $m$  7's come from  $[3, 4, 7]$ . But then the tableau cannot have a  $[2, 5, 7]$  otherwise semistandardness will be violated. So we only have  $2m$  columns of  $f$  with  $[3, 5, 7]$ . However this means we have  $3m + 1$  3's, a contradiction. So we may assume that we have a  $[2, 4, 7]$ . We show then that there are at least two  $[3, 5, 7]$ 's, so the tableau indexing  $y_3$  is a subtableau. If we have only one  $[3, 5, 7]$  we would have  $2m - 1$   $[2, 5, 7]$ 's. But we already have one  $[2, 4, 7]$ , one  $[1, 2, 5]$  and  $m$   $[2, 4, 6]$ 's, a contradiction to the number of allowed 2's.

d If the first column of  $f$  is  $[1, 3, 5]$ . Then all the 2's occur in the first row of  $f$  and in columns  $3m + 1$  to  $6m$  and there are  $m$  3's in the last  $m$  columns of the first row. Since there are no 3's in the last row, the remaining  $2m$  3's must occur in the second row. Since 6 occurs in the last row in positions  $m + 1$  to  $4m$ , and a 1 occurs in the first row in columns 1 to  $3m$ , it follows that  $[1, 3, 6]$  is a column in  $f$ . Now all the 4's occur in the second row, starting at position  $2m + 1$  and ending at position  $5m$ , after which we only have 5's in the second row. Since the 2's in the first row of  $f$  occur in positions  $3m + 1$  to  $6m$  and the 7's occur in the bottom row in position  $4m + 1$  to  $7m$  it follows that there is a column containing  $[2, 4, 7]$  and a column containing  $[2, 5, 7]$ . So  $y_1$  is a subtableau of  $f$ .

e In case the first column is  $[1, 3, 4]$ , all the 2's occur in the first row, and so we have  $m$  3's in the first row appearing in the columns  $[3, 4, 7]$  and  $[3, 5, 7]$ . So  $f$  has at least  $m$  columns with  $[2, 5, 7]$ . Now there are  $2m$  3's in the second row and these must occur in positions 1 to  $2m$ . Since the last row has 6 in columns  $m + 1$  to  $4m$  and the first row has a 1 in columns 1 to  $3m$ , it follows that there is a column with filling  $[1, 3, 6]$  in the given tableau. It follows that the tableau indexing  $y_2$  is a subtableau of  $f$ . We are done by induction.

□

**Remark 4.6.** Let  $\tau_1 = [2, 5, 7]$ ,  $\tau_2 = [3, 4, 7]$ ,  $\tau_3 = [2, 4, 7]$ ,  $\tau_4 = [3, 5, 7]$ ,  $\tau_5 = [2, 3, 7]$ ,  $\tau_6 = [4, 5, 7]$ . Consider the product of the Plücker coordinates  $p_{\tau_1}p_{\tau_2}$ . The straightening law gives us  $p_{\tau_1}p_{\tau_2} = p_{\tau_3}p_{\tau_4} - p_{\tau_5}p_{\tau_6}$ . On the Schubert variety  $X(w_{3,7})$ ,  $p_{\tau_6} = 0$ . So, on  $X(w_{3,7})$ ,  $p_{\tau_1}p_{\tau_2} = p_{\tau_3}p_{\tau_4}$ . As a result  $y_5y_7 = z$ .

**Theorem 4.7.** The ring  $R = \bigoplus_{m \geq 0} H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$  is generated in degree 1.

*Proof.* We continue to use the notation  $y_1, \dots, y_1, z$  for the semistandard tableau basis elements and the monomials they index.

The proof is by induction on  $m$ , the base case  $m = 1$  being obvious. Now assume  $m \geq 2$ . Given any semistandard basis element of  $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$ , Lemma 4.5 shows that it can be written as a product of one of the  $y_i$ 's and a semistandard basis element of  $R(m-1)$ , or as a product of  $z$  in  $R(2)$ , and a semistandard basis element of  $R(m-2)$ . Because of Remark 4.6 we have  $z = y_5y_7$ . So we can replace  $z$  by  $y_5y_7$ . It follows by induction that every basis element of  $R(m)$  is in the algebra generated by the  $y_i$ 's. □

It follows that there is a surjective ring homomorphism  $\Phi : \mathbb{C}[Y_1, Y_2, \dots, Y_7] \rightarrow R$ , sending  $Y_i$  to  $y_i$ .

Now let  $\mathcal{I}$  be the two sided ideal generated by the following relations in  $\mathbb{C}[Y_1, Y_2, \dots, Y_7]$ .

$$Y_1Y_4 = Y_2Y_3 - Y_2Y_7 + Y_1Y_7, \tag{2a}$$

$$Y_1Y_5 = Y_3^2 - Y_3Y_7, \tag{2b}$$

$$Y_1Y_6 = Y_3Y_4 - Y_4Y_7, \tag{2c}$$

$$Y_2Y_5 = Y_3Y_4 - Y_3Y_7, \tag{2d}$$

$$Y_2Y_6 = Y_4^2 - Y_4Y_7, \tag{2e}$$

$$Y_3Y_6 = Y_4Y_5. \tag{2f}$$

**Theorem 4.8.** The map  $\Phi$  induces an isomorphism  $\tilde{\Phi} : \mathbb{C}[Y_1, Y_2, \dots, Y_7]/\mathcal{I} \simeq R$ .

*Proof.* By explicit calculations one can check that the above relations hold with  $Y_i$  replaced by  $y_i$ ; they are in the kernel of  $\tilde{\Phi}$ . We omit these calculations. To complete the proof we show we can use the above relations as a reduction system. The process consists of replacing a monomial  $M$  in the  $Y_i$ 's which is divisible by a term  $L_i$  on the left hand side of one of the reduction rules  $L_i = R_i$ , by  $(M/L_i)R_i$ . Here  $R_i$  is the right hand side of  $L_i = R_i$ .

We show that the diamond lemma of ring theory holds for this reduction system (see [1]). What this implies is that any monomial in the  $Y_i$ 's reduces, after applying these reductions (in any order, when multiple reduction rules apply) to a unique expression in the  $Y_i$ 's, in which no term is divisible by a term appearing on the left hand side of the above reduction system.

We prove that the diamond lemma holds for this reduction system by looking at the reduction of the minimal overlapping ambiguities  $Y_1Y_2Y_5, Y_1Y_2Y_6, Y_1Y_3Y_6$  and  $Y_2Y_3Y_6$ . We show in each case that the final expression is unambiguous. It follows that any relation among  $Y_i$ 's is in the two sided ideal  $\mathcal{I}$  generated by the above relations. This proves that the map  $\tilde{\Phi}$  constructed above is injective.

To complete the proof we look at the reductions of overlapping ambiguities.

$Y_1Y_2Y_5$  - using rule (2b) above we get  $Y_2(Y_3^2 - Y_3Y_7) = Y_2Y_3^2 - Y_2Y_3Y_7$ , which cannot be reduced further. On the other hand using rule (2d) above we get  $Y_1(Y_3Y_4 - Y_3Y_7)$ . Now this can be further reduced using rule (2a) and we get  $Y_3Y_1Y_7 + Y_3Y_2Y_3 - Y_3Y_2Y_7 - Y_1Y_3Y_7$ , and this is equal to  $Y_2Y_3^2 - Y_2Y_3Y_7$ . The reduction is unique in this case.

Likewise, one can show that  $Y_1Y_2Y_6$  reduces to the unique expression  $Y_2Y_3Y_4 - Y_2Y_4Y_7$ ,  $Y_2Y_5Y_6$  reduces to  $Y_4^2Y_5 - Y_4Y_5Y_7$  and  $Y_2Y_3Y_6$  reduces to  $Y_3Y_4^2 - Y_3^2Y_4Y_7$ , completing the proof.  $\square$

## 5 Deodhar decomposition to compute quotients of Richardson varieties

This section is again motivated by the question of understanding GIT quotients of Richardson varieties in the Grassmannian. In Section 3 we proved some results on quotients of Richardson varieties. A natural strategy to understand the GIT quotient is to take a stratification of a Richardson variety, understand what the GIT quotient of each strata is, and also understand how the GIT quotients of these strata patch up. Such a stratification of the open cell of a Richardson variety was given by Deodhar [3]. This was to be our starting point. Working with small examples we believed that the restriction of a  $T$ -invariant section to the open cell would be a homogenous polynomial and that this would lead us to discover the equations defining the GIT quotient of a Richardson variety. However we soon realized that sections may not restrict to homogenous polynomials on the open cell, and that the issue is more subtle. We have necessary conditions which guarantee when sections restrict to homogenous polynomials on the open cell. This is Lemma 5.13. To state the Lemma and also the proof we need to introduce the Deodhar decomposition and some more notation and theorems about Deodhar decomposition of Richardson varieties on the Grassmannian. We do that in the next Subsection 5.1. We use the Deodhar decomposition to study the GIT quotients of Richardson varieties in  $X(w_{3,7})$  in Section 5.2. Although all these results follow from the results in Section 3 we prove them again since this can be done by explicit calculations. Finally we show that the GIT quotient of  $X(w_{3,7})$  is a rational normal scroll. We were unable to complete this proof using only information about the GIT quotients of Richardson varieties in  $X(w_{3,7})$ . Instead we show that the equations defining the GIT quotient is a determinantal variety.



## 5.1 Deodhar decomposition

In [3] Deodhar considered the intersection in  $G/B$  of the open cell in a Schubert variety with the open cell of an opposite Schubert variety. For  $v, w \in W$ , define the Richardson strata<sup>3</sup>

$$R_w^v = (BwB/B) \cap (B^-vB/B).$$

Note that this is not the same as the definition of a Richardson variety (see [2]). Recall that for  $v, w \in W$ , a Richardson variety in  $G/B$  was defined to be the intersection of  $X(w) \cap X^v$ . Since both  $X(w)$  and  $X^v$  contain the intersection of  $(BwB/B) \cap (B^-vB/B)$  it is clear that  $R_w^v \subseteq X_w^v$ . And so the Richardson strata is empty if  $v \not\leq w$  and the closure of  $R_w^v$  is  $X_w^v$ .

In [3] Deodhar gave a refined decomposition of a Richardson strata in  $G/B$  into disjoint, locally closed subvarieties of a Schubert variety. We follow the notation from Marsh and Rietsch [23], and Kodama and Williams [20]. The definitions and examples are taken verbatim from [20] since it is their notation and set up that we use in our proofs.

Fix a reduced decomposition  $\mathbf{w} = s_{i_1}s_{i_2}\cdots s_{i_m}$ . We define a subexpression  $\mathbf{v}$  of  $\mathbf{w}$  to be a word obtained from the reduced expression  $\mathbf{w}$  by replacing some of the factors with 1. For example, consider a reduced expression in  $S_4$ , say  $s_3s_2s_1s_3s_2s_3$ . Then  $s_3s_21s_3s_21$  is a subexpression of  $s_3s_2s_1s_3s_2s_3$ . Given a subexpression  $\mathbf{v}$ , we set  $\mathbf{v}_{(k)}$  to be the product of the leftmost  $k$  factors of  $\mathbf{v}$  if  $k \geq 1$ , and set  $\mathbf{v}_{(0)} = 1$ . The following definition was given in [23] and was inspired from Deodhar's paper [3].

**Definition 5.1.** *Given a subexpression  $\mathbf{v}$  of a reduced expression  $\mathbf{w} = s_{i_1}s_{i_2}\cdots s_{i_m}$ , we define*

$$\begin{aligned} J_{\mathbf{v}}^{\circ} &:= \{k \in \{1, \dots, m\} \mid \mathbf{v}_{(k-1)} < \mathbf{v}_{(k)}\}, \\ J_{\mathbf{v}}^{\square} &:= \{k \in \{1, \dots, m\} \mid \mathbf{v}_{(k-1)} = \mathbf{v}_{(k)}\}, \\ J_{\mathbf{v}}^{\bullet} &:= \{k \in \{1, \dots, m\} \mid \mathbf{v}_{(k-1)} > \mathbf{v}_{(k)}\}. \end{aligned}$$

The expression  $\mathbf{v}$  is called non-decreasing if  $\mathbf{v}_{(j-1)} \leq \mathbf{v}_{(j)}$  for all  $j = 1, \dots, m$ , and in this case  $J_{\mathbf{v}}^{\bullet} = \emptyset$ .

The following is from [3, Definition 2.3].

**Definition 5.2.** *(Distinguished subexpressions). A subexpression  $\mathbf{v}$  of  $\mathbf{w}$  is called distinguished if we have*

$$\mathbf{v}_{(j)} \leq \mathbf{v}_{(j-1)}s_{i_j}, \quad \forall j \in \{1, \dots, m\}.$$

In other words, if right multiplication by  $s_{i_j}$  decreases the length of  $\mathbf{v}_{(j-1)}$ , then in a distinguished subexpression we must have  $\mathbf{v}_{(j)} = \mathbf{v}_{(j-1)}s_{i_j}$ .

We write  $\mathbf{v} \prec \mathbf{w}$  if  $\mathbf{v}$  is a distinguished subexpression of  $\mathbf{w}$ .

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<sup>3</sup>This terminology is not standard. What we have called strata is sometimes called a Richardson variety

**Definition 5.3.** (*Positive distinguished subexpressions*). We call a subexpression  $\mathbf{v}$  of  $\mathbf{w}$  a positive distinguished subexpression (or a PDS for short) if  $\mathbf{v}_{(j-1)} < \mathbf{v}_{(j-1)}s_j$ , for all  $j \in \{1, \dots, m\}$ .

Reitsch and Marsh [23] proved the following lemma.

**Lemma 5.4.** *Given  $v \leq w$  and a reduced expression  $\mathbf{w} = s_{i_1} \cdots s_{i_m}$  for  $w$ , there is a unique PDS  $\mathbf{v}^+$  for  $v$  in  $\mathbf{w}$ .*

We now describe the Deodhar decomposition of the Richardson strata. Marsh and Rietsch [23] gave explicit parameterizations for each Deodhar component, identifying each one with a subset in the group. Much of this appears implicitly in Deodhar's paper, but we refer to [23] for our exposition because these statements are made explicit there and the authors make references to Deodhar's paper wherever needed.

**Definition 5.5.** [23, Definition 5.1] Let  $\mathbf{w} = s_{i_1} \cdots s_{i_m}$  be a reduced expression for  $w$ , and let  $\mathbf{v}$  be a distinguished subexpression. Define a subset  $\mathbf{G}_{\mathbf{w}}^{\mathbf{v}}$  in  $G$  by

$$\mathbf{G}_{\mathbf{w}}^{\mathbf{v}} := \left\{ g = g_1 g_2 \cdots g_m \mid \begin{cases} g_l = x_{i_l}(m_l) s_{i_l} & \text{if } l \in J_{\mathbf{v}}^{\bullet}, \\ g_l = y_{i_l}(p_l) & \text{if } l \in J_{\mathbf{v}}^{\square}, \\ g_l = s_{i_l} & \text{if } l \in J_{\mathbf{v}}^{\circ} \end{cases} \right\}$$

In the above definition the matrices  $x_{i_l}(m_l), y_{i_l}(p_l), s_{i_l}$  are as defined in Equation 1, Section 2. From [23, Theorem 4.2] there is an isomorphism from  $\mathbb{C}^{*|J_{\mathbf{v}}^{\square}|} \times \mathbb{C}^{|J_{\mathbf{v}}^{\bullet}|}$  to  $\mathbf{G}_{\mathbf{w}}^{\mathbf{v}}$ .

**Definition 5.6.** (*Deodhar Component*) The Deodhar component  $\mathcal{R}_{\mathbf{w}}^{\mathbf{v}}$  is the image of  $\mathbf{G}_{\mathbf{w}}^{\mathbf{v}}$  under the map  $\mathbf{G}_{\mathbf{w}}^{\mathbf{v}} \subseteq U^{-}vB \cap BwB \rightarrow G/B$ , sending  $g$  to  $gB$ .

Then from [3, Theorem 1.1] one has [3, Corollary 1.2], also from Deodhar.

**Theorem 5.7.**  $R_w^v = \bigsqcup_{\mathbf{v} \prec \mathbf{w}} \mathcal{R}_{\mathbf{w}}^{\mathbf{v}}$  the union taken over all distinguished subexpressions  $\mathbf{v}$  such that  $\mathbf{v}_{(m)} = v$ . The component  $\mathcal{R}_{\mathbf{w}}^{\mathbf{v}^+}$  is open in  $R_w^v$ .

The Deodhar decomposition of a Richardson strata in  $G/P_{\hat{\alpha}_r}$  is obtained by taking the projection of the Richardson strata in  $G/B$  to  $G/P_{\hat{\alpha}_r}$ . In [20, Proposition 4.16] the authors show that the Deodhar components of a Richardson strata in  $G/P_{\hat{\alpha}_r}$  are independent of  $\mathbf{w}$  and only depends upon  $w$ . This follows from the observation that any two reduced decompositions  $\mathbf{w}$  and  $\mathbf{w}'$  of  $w$  are related by a sequence of commuting transpositions  $s_i s_j = s_j s_i$ .

## 5.2 Quotients of Deodhar components in $X(w_{3,7})$

Let us fix a reduced decomposition  $\mathbf{w}_{3,7} = s_2 s_1 s_4 s_3 s_6 s_5 s_2 s_4 s_3$  for the Weyl group element  $w_{3,7}$  with  $X(w_{3,7})$  being the minimal Schubert variety in  $G_{3,7}$  admitting semistable points. In

this section we describe the GIT quotients of Richardson varieties in  $X(w_{3,7})$  by computing the various Deodhar strata in this Schubert variety and analyzing their quotients. It will be useful to recall Definition 5.5 and the notation developed in Subsection 5.1.

We begin with a corollary to Theorem 4.1.

**Corollary 5.8.** *The GIT quotient of Richardson varieties in  $X(w_{3,7})$  is projectively normal with respect to the descent of the  $T$ -linearized line bundle  $\mathcal{L}(7\omega_3)$ .*

*Proof.* Let  $X_{w_{3,7}}^v$  be a Richardson variety in  $X(w_{3,7})$ . From the proof of [2, Proposition 1 ] it follows that  $H^0(X(w_{3,7}), \mathcal{L}(\omega_3)^{\otimes m}) \rightarrow H^0(X_{w_{3,7}}^v, \mathcal{L}(\omega_3)^{\otimes m})$  is surjective. Since  $T$  is linearly reductive it follows that the map  $H^0(X(w_{3,7}), \mathcal{L}(\omega_3)^{\otimes m})^T \rightarrow H^0(X_{w_{3,7}}^v, \mathcal{L}(\omega_3)^{\otimes m})^T$  is also surjective. From Theorem 4.1 we know that the polarized variety  $(T \backslash\backslash X(w_{3,7})_T^{ss}(\mathcal{L}(7\omega_3)), \tilde{\mathcal{L}}(7\omega_3))$  is projectively normal. Since  $T \backslash\backslash (X_{w_{3,7}}^v)_T^{ss}(\mathcal{L}(7\omega_3))$  is normal it follows that the GIT quotient of  $X_{w_{3,7}}^v$  is projectively normal with respect to the descent line bundle.  $\square$

**Lemma 5.9.** *Let  $v = s_2s_4s_3$ . Then  $T \backslash\backslash (X_{w_{3,7}}^v)_T^{ss}(\mathcal{L}(7\omega_3))$  is a point.*

*Proof.* The only torus-invariant section of  $H^0(X(w_{3,7}), \mathcal{L}(7\omega_3))$  which is non-zero on  $X_{w_{3,7}}^v$  is the section  $y_1$ . Consider the Deodhar component of  $X_{w_{3,7}}^v$  corresponding to the subexpression  $\mathbf{v} = 11111s_2s_4s_3$ . From Definition 5.5, every matrix in  $\mathbf{G}_{\mathbf{w}_{3,7}}^{\mathbf{v}}$  is a product of matrices  $y_2(p_1)y_1(p_2)y_4(p_3)y_3(p_4)y_6(p_5)y_5(p_6)s_2s_4s_3$ . We use Equation 1, Section 2 to obtain each term of the product. Multiplying these terms we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ p_1p_2 & 1 & 0 & p_1 & 0 & 0 & 0 \\ 0 & p_4 & 0 & 0 & 1 & 0 & 0 \\ 0 & p_3p_4 & 1 & 0 & p_3 & 0 & 0 \\ 0 & 0 & p_6 & 0 & 0 & 1 & 0 \\ 0 & 0 & p_5p_6 & 0 & 0 & p_5 & 1 \end{pmatrix}$$

The corresponding point in  $G_{3,7}$  is the three dimensional subspace spanned by the first three columns of the above matrix. Denote this submatrix by  $M$ . The Plücker coordinates of the embedding of  $M$  in projective space are the determinants of the the  $3 \times 3$  submatrices of  $M$ . The section  $y_1$  evaluated on such a matrix  $M$  in  $\mathbf{G}_{\mathbf{w}_{3,7}}^{\mathbf{v}}$  is the product of the  $3 \times 3$  determinants of  $M$  whose rows are indexed by the columns in  $y_1$ . For the above matrix this evaluates to  $p_1p_2^4p_3^2p_4^5p_5^3p_6^6$ . Using the reduced expression for  $\mathbf{w}_{3,7}$ , note that the weight of this monomial is  $\alpha_2 + 4\alpha_1 + 2\alpha_4 + 5\alpha_3 + 3\alpha_6 + 6\alpha_5$ .  $\square$

**Lemma 5.10.** *Let  $v = s_2s_3$ . Then  $T \backslash\backslash (X_{w_{3,7}}^v)_T^{ss}(\mathcal{L}(7\omega_3))$  is isomorphic to  $\mathbb{P}^1$  and the descent of  $\mathcal{L}(7\omega_3)$  is  $\mathcal{O}(1)$ .*

*Proof.* On the open Deodhar component corresponding to the reduced subexpression  $\mathbf{v} = 11111s_21s_3$  the only non-zero  $T$ -invariant standard monomials of shape  $7\omega_3$  are  $y_1, y_2$ . Using

a calculation as in the proof of Lemma 5.9 we see that these are algebraically independent. The lemma follows now from Corollary 5.8.  $\square$

**Lemma 5.11.** *Let  $v = s_4s_3$ . Then  $T \backslash \backslash (X_{w_{3,7}}^v)_T^{ss}(\mathcal{L}(7\omega_3))$  is isomorphic to  $\mathbb{P}^1$  and the descent of  $\mathcal{L}(7\omega_3)$  to the GIT quotient is  $\mathcal{O}(2)$ .*

*Proof.* The three non-zero sections on the open Deodhar cell  $\mathbf{G}_{w_{3,7}}^v$  corresponding to the subexpression  $\mathbf{v} = 111111s_4s_3$  are  $y_1, y_3, y_5$ . Let  $p = p_1p_2^4p_3^2p_4^5p_5^3p_6^6p_7^5$ . Let  $X = (p_1 + p_7), Y = p_1$ . Calculating as in the proof of Lemma 5.9, one checks that on the open Deodhar cell  $\mathbf{G}_{w_{3,7}}^v$ ,  $y_1$  evaluates to  $pX^2$ ,  $y_3$  to  $pXY$  and  $y_5$  to  $pY^2$ . So  $p$  is nowhere vanishing on the Deodhar cell. The lemma follows from Corollary 5.8.  $\square$

**Lemma 5.12.** *Let  $v = s_3$ . Then  $T \backslash \backslash (X_{w_{3,7}}^v)_T^{ss}(\mathcal{L}(7\omega_3))$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  and the descent of the line bundle to the GIT quotient is  $\mathcal{O}(2) \boxtimes \mathcal{O}(1)$ .*

*Proof.* We use the distinguished subexpression  $\mathbf{v} = 1111111s_3$ . Let  $A = p_3, B = p_3 + p_8$ . Let  $X = (p_1 + p_7), Y = p_1$ . Note that  $p_3$  and  $p_8$  are algebraically independent and so  $A, B$  are algebraically independent. Since  $p_1$  and  $p_7$  are algebraically independent so are  $X, Y$ .

Let  $p = p_1p_2^4p_3^2p_4^5p_5^3p_6^6p_7^5p_8^6$ . Calculating as in the proof of Lemma 5.9, we see that  $y_1$  evaluates to  $pBX^2$  on  $\mathbf{G}_{w_{3,7}}^v$ . The section  $y_5$  evaluates to  $pBY^2$ ,  $y_3$  evaluates to  $pXYB$ ,  $y_2$  evaluates to  $pAX^2$ ,  $y_6$  evaluates to  $pAY^2$  and  $y_4$  evaluates to  $pXYA$ .

So upto a multiple of  $p$ , the sections  $y_2, y_4, y_6, y_1, y_3, y_5$  can be respectively written as  $(X^2A, XYA, Y^2A, X^2B, XYB, Y^2B)$ . Using Corollary 5.8 it follows that the GIT quotient is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  embedded as  $\mathcal{O}(2) \boxtimes \mathcal{O}(1)$ .  $\square$

In the next lemma we give conditions guaranteeing when a section of the line bundle  $\mathcal{L}(n\omega_r)$  on  $X_w^v$  restricts to a homogenous polynomial on the Richardson strata in  $X_w^v$ .

**Lemma 5.13.** *Let  $u \in W, v \in W^{S \setminus \{\alpha_r\}}$  be such that  $w = uv \in W^{S \setminus \{\alpha_r\}}$  and  $l(uv) = l(u) + l(v)$ . Fix a reduced expression  $u = s_{i_1} \cdots s_{i_k}$  and a reduced expression  $v = s_{i_{k+1}} \cdots s_{i_m}$  such that  $\mathbf{w} = s_{i_1} \cdots s_{i_k} s_{i_{k+1}} \cdots s_{i_m}$  is a reduced expression for  $w$ . Consider  $\mathbf{v} = 1 \cdots 1 s_{i_{k+1}} \cdots s_{i_m}$ , a distinguished subexpression of  $\mathbf{w}$ .  $\mathcal{R}_{\mathbf{w}}^v$  is the unique open Deodhar component of  $R_w^v$ . The restriction of any section  $s \in H^0(X_w^v, \mathcal{L}(n\omega_r))$  to  $R_w^v$  is a homogeneous polynomial in  $p_1, p_2, \dots, p_k$  having degree equal to the height of  $v(n\omega_r)$ .*

*Proof.* Note that  $\mathbf{v}$  is the unique positive distinguished subexpression for  $v$  in  $\mathbf{w}$  and so  $\mathcal{R}_{\mathbf{w}}^v$  is the unique open Deodhar component of  $R_w^v$ .

Matrices in  $\mathbf{G}_{\mathbf{w}}^v$  are of the form  $y_{i_1}(p_1)y_{i_2}(p_2) \cdots y_{i_k}(p_k)s_{i_{k+1}} \cdots s_{i_m}$ . From this identification we see that the section  $s$  restricted to this Deodhar component is  $s|_{\mathcal{R}_{\mathbf{w}}^v} = \sum_{\mathbf{m}} a_{\mathbf{m}} p_1^{m_1} \cdots p_k^{m_k}$  where  $\mathbf{m} = (m_1, \dots, m_k)$ . If  $a_{\mathbf{m}} \neq 0$  then  $wt(s) = wt(p_1^{m_1} \cdots p_k^{m_k}) = v(n\omega_r)$ . In particular,  $deg(p_1^{m_1} \cdots p_k^{m_k})$  is equal to the height of  $v(n\omega_r)$ .  $\square$

Finally we prove the main theorem of this section.

**Theorem 5.14.** *The polarized variety  $(T \backslash (X_{w_{3,7}}^{id})_{T}^{ss}(\mathcal{L}(7\omega_3)), \tilde{\mathcal{L}}(7\omega_3))$  is a rational normal scroll.*

*Proof.* The relations (2a)-(2f) given before Theorem 4.8 describe the homogenous ideal defining the polarized variety. These defining relations can be written succinctly in a matrix form

$$\text{rank} \begin{pmatrix} Y_1 & Y_3 & Y_4 & Y_2 \\ Y_3 - Y_7 & Y_5 & Y_6 & Y_4 - Y_7 \end{pmatrix} \leq 1.$$

For example the minor corresponding to the first two columns above gives us  $Y_1Y_5 = Y_3^2 - Y_3Y_7$ , which is (2b), and the minor corresponding to columns 1 and 3 gives relation (2c) shown there. The polarized variety is defined by the vanishing of  $2 \times 2$  determinants of a generic  $4 \times 2$  matrix. Such a variety is called a determinantal variety and it is known that it is a rational normal scroll (see [24]).  $\square$

## 6 Projective normality of the GIT quotient of $G_{2,n}$

In this section we study the GIT quotient of  $G_{2,n}$  with respect to the  $T$ -linearized line bundle  $\mathcal{L}(n\omega_2)$  for  $n$  odd. As mentioned earlier, this line bundle descends to the quotient, and it is well known that the polarized variety  $((T \backslash (G_{2,n}))_{T}^{ss}(\mathcal{L}(n\omega_2)), \tilde{\mathcal{L}}(n\omega_2))$  is projectively normal (see [9], [19]). It is not clear to us whether this result extends to GIT quotients of higher rank Grassmannians. To the best of our knowledge this question is open. We give an alternate combinatorial proof in the  $G_{2,n}$  case. We believe that it is this kind of combinatorics which will be required to settle the general question.

We follow the strategy outlined in Remark 4.2. Defining  $R(m)$  to be  $H^0(G_{2,n}, \mathcal{L}(n\omega_2)^{\otimes m})^T$  we show that  $R(1)^{\otimes m} \rightarrow R(m)$  is surjective.

Let  $p_{\underline{\tau}} = p_{\tau_1}p_{\tau_2} \dots p_{\tau_{mn}}$  be a standard monomial in  $R(m)$  and let  $T_{\underline{\tau}}$  be the tableau associated to this monomial.

Denote the columns of  $T_{\underline{\tau}}$  by  $C_1, C_2, \dots, C_{mn}$  with  $C_i = [a_i, b_i]$ . Our idea is to extract from the tableau  $T_{\underline{\tau}}$  a semistandard Young subtableau  $T_{\underline{\mu}}$ , with each integer  $1, 2, \dots, n$  appearing exactly 2 times. Then the monomial  $p_{\underline{\mu}}$  corresponding to this subtableau would be a zero weight vector in  $R(1)$ , and the monomial corresponding to the remaining columns in  $T_{\underline{\tau}}$  would be a monomial  $p_{\underline{\nu}} \in R(m-1)$ . If we were to succeed in doing this, we could write  $p_{\underline{\tau}}$  as a product of  $p_{\underline{\mu}}$  and  $p_{\underline{\nu}}$ , and we would be done by induction on  $m$ . Since we were unable to do this directly we use straightening laws on tableaux to show that  $p_{\underline{\tau}}$  can be written as a sum of products of elements in  $R(1)$ .

Let  $p_{\underline{\mu}} := p_{\tau_1}p_{\tau_{m+1}} \dots p_{\tau_{mn-m+1}}$  and  $p_{\underline{\nu}} = \widehat{p_{\tau_1}}p_{\tau_2}p_{\tau_3} \dots p_{\tau_m}\widehat{p_{\tau_{m+1}}} \dots p_{\tau_{mn}}$ . Here  $\widehat{p}$  indicates that the corresponding term is omitted. Clearly  $p_{\underline{\tau}} = p_{\underline{\mu}}p_{\underline{\nu}}$ .

Let  $T_{\underline{\mu}}$  and  $T_{\underline{\nu}}$  denote the corresponding tableaux.

**Definition 6.1.** An integer  $i$  is defected if  $i$  appears an odd number of times in  $T_{\underline{\mu}}$ . Denote the set of defected integers by  $\mathcal{D}$ .

**Lemma 6.2.** All integers in  $\{1, 2, \dots, n\}$  occur in  $T_{\underline{\mu}}$ .

*Proof.* Every integer  $j$  has to appear at least  $m$  times in one of the rows of  $T_{\underline{\tau}}$ . Since  $T_{\underline{\tau}}$  is semistandard  $j$  appears consecutively in that row, so there is a column  $C_i$  with  $i \equiv 1 \pmod{m}$  containing  $j$ .  $\square$

**Lemma 6.3.** There are even number of defected integers.

*Proof.*  $T_{\underline{\mu}}$  has  $2n$  boxes and all the integers appear in  $T_{\underline{\mu}}$ . Each integer which is not defected appears twice. The number of times a defected integer appears is odd, so there are an even number of defected integers.  $\square$

Before we prove the next lemma we set up some notation and make some observations.

Let  $f_i$  (respectively,  $l_i$ ) be such that  $C_{f_i}$  (respectively,  $C_{l_i}$ ) is the column in which  $i$  appears for the first (respectively, last) time in the bottom row of  $T_{\underline{\tau}}$ . Similarly define  $f^i$  and  $l^i$  with respect to occurrences of  $i$  in the top row.

**Observation 6.4.**  $f_i \equiv x+1 \pmod{m}$  if and only if  $f^i \equiv m-x+1 \pmod{m}$ . In particular  $f_i \equiv 1 \pmod{m}$  if and only if  $f^i \equiv 1 \pmod{m}$ , and in this case  $i$  appears at least 2 times in  $T_{\underline{\mu}}$ .

*Proof.* Each integer less than  $i$  appears  $2m$  times and occurs in the top row in columns before column  $f^i$  and in the bottom row in columns before column  $f_i$ . The total number of positions for numbers from 1 to  $i-1$  is therefore a multiple of  $m$ . If  $f_i$  is  $am+1+x$ , then the number of boxes to the left of this column in the bottom row is  $am+x$ . So  $f^i$  must be  $bm+m-x+1$  for some  $b$  so that the number of positions for integers 1 to  $i-1$  is  $bm+m-x$  as needed.

The last statement follows since  $T_{\underline{\mu}}$  is constructed by taking only columns numbered  $1 \pmod{m}$  in  $T_{\underline{\tau}}$   $\square$

We know that the number of defected integers is an even number, say  $2l$ , for  $1 \leq l \leq \lfloor \frac{n}{2} \rfloor$ . Let  $\mathcal{D} = \{i_1, i_2, \dots, i_{2l}\}$  denote this set with  $i_1 < i_2 < \dots < i_{2l}$ .

**Lemma 6.5.** Let  $1 \leq j \leq 2l$ . In  $T_{\underline{\mu}}$ ,  $i_j$  appears 3 times if  $j$  is odd and  $i_j$  appears once if  $j$  is even.

*Proof.* We show that two consecutive defected integers cannot appear in  $T_{\underline{\mu}}$  3 times and they cannot appear once. Then we show that the first integer which is defected appears 3 times.

Let us assume that some integer  $i_j$  which is defected appears 3 times. Without loss of generality we may assume that it appears 2 times in the top row and appears once in the

bottom row. Assume that the next defected integer  $i_{j+1}$  also appears 3 times. We prove it in the case when  $i_{j+1}$  appears 2 times in the top row and once in the bottom row. The proof in the other case is similar.

Assume that the positions of  $i_j$  (respectively,  $i_{j+1}$ ) in  $T_{\underline{\tau}}$  which contribute to its two occurrences in the top row of  $T_{\underline{\mu}}$  are  $(a-1)m+1, am+1$  (respectively,  $bm+1, (b+1)m+1$ ). Likewise, assume that the positions of  $i_j$ , (respectively,  $i_{j+1}$ ) in  $T_{\underline{\tau}}$  contributing to the bottom row in  $T_{\underline{\mu}}$  are  $cm+1$  (respectively,  $dm+1$ ). Clearly  $c < a-1$  and  $d < b$ . Let  $x$  be the number of  $i_j$  to the right of position  $am+1$  in the top row of  $T_{\underline{\tau}}$  and  $z$  be the number of  $i_j$  to the right of  $cm+1$  in the bottom row of  $T_{\underline{\tau}}$ . Similarly let  $y$  denote the number of  $i_{j+1}$  to the left of position  $bm+1$  in the top row and  $w$  be the number of  $i_{j+1}$  to the left of position  $dm+1$  in the bottom row of  $T_{\underline{\tau}}$ . Clearly  $x+z \leq m-2$  and  $y+w \leq m-2$ .

Now  $i_{j+1} = i_j + 1$  is not possible. Because the number of  $i_{j+1}$  in the top row is then at least  $2m-x$  and the number of  $i_{j+1}$  in the bottom row is at least  $m-z$ , a contradiction to the number of  $i_{j+1}$  in  $T_{\underline{\tau}}$  since  $x+z \leq m-2$ .

So let us assume that  $i_{j+1} > i_j + 1$ . Now there are  $i_{j+1} - i_j - 1$  integers in between  $i_j$  and  $i_{j+1}$  which are not defected. Hence in  $T_{\underline{\tau}}$  each of these integers occurs in exactly two positions which are in positions  $1 \pmod{m}$ . Hence the number of positions which are  $1 \pmod{m}$  between the positions  $am+1, bm+1$  and between  $cm+1, dm+1$  is exactly  $2(i_{j+1} - i_j - 1)$ . But this count is also equal to  $(b-a-1) + (d-c-1)$ . Hence  $b+d-a-c-2 = 2(i_{j+1} - i_j - 1)$ . Or  $b+d-a-c = 2(i_{j+1} - i_j)$ . The total number of positions available for integers in the range  $i_j + 1$  to  $i_{j+1}$  is exactly  $bm - am - x - y - 1 + dm - cm - z - w - 1$  which is  $m(b+d-a-c-2) - (x+y+z+w)$ . Since each integer in this range appears exactly  $2m$  times, and since  $b+d-a-c = 2((i_{j+1} - i_j))$  it follows that  $x+y+z+w$  is 0 modulo  $2m$ . Since  $x+z \leq m-2$  and  $y+w \leq m-2$  this is only possible if  $x, y, z, w$  are all zero. Then the positions  $am+1+1$  to  $bm$  and  $cm+1+1$  to  $dm$  are available for the integers  $i_j+1, \dots, i_{j+1}-1$ . This is  $(b+d-a-c)m-2$  positions in all, which is also  $2(i_{j+1} - i_j)m-2$  positions. But this is more positions than are required, since we have  $i_{j+1} - i_j - 1$  numbers each occurring  $2m$  times - we require only  $2m(i_{j+1} - i_j - 1)$  positions. We conclude that if  $i_j$  appears 3 times  $i_{j+1}$  cannot appear 3 times.

Now we show that if  $i_j$  appears with defect one then  $i_{j+1}$  appears with defect three. Without loss of generality assume that  $i_j$  appears in the top row in a column numbered  $1 \pmod{m}$ . So we know that  $i_j$  appears less than  $m$  times in the bottom row.

Assume that  $f^{i_j}$  is  $am+x+1$  for some  $1 \leq x \leq m-1$ . Then  $f_{i_j}$  is  $bm+m-x+1$  for some  $b$ . Now since  $i_j$  does not occur in a column numbered  $1 \pmod{m}$  in the bottom row, it follows that the number of  $i_j$  in the bottom is at most  $x$ , so the number of  $i_j$  in the top row is at least  $2m-x$ . But since there is only one occurrence of  $i_j$  in a column numbered  $1 \pmod{m}$ , there are at most  $2m-x$  occurrences of  $i_j$  in the top row. It follows that the top row has exactly  $2m-x$  occurrences of  $i_j$  and the bottom row has exactly  $x$  occurrences of  $i_j$ . So  $l^{i_j} = l_{i_j} = 0 \pmod{m}$ . Hence each integers between  $i_j$  and  $i_{j+1}$  which is not defected starts at a position which is  $1 \pmod{m}$  on the top and ends at a  $0 \pmod{m}$  position in

the top and bottom rows (if it occurs in them). So  $f_{i_{j+1}}$  is forced to be  $1 \pmod{m}$  and this implies that  $f^{i_{j+1}}$  is also  $1 \pmod{m}$  by Observation 6.4. Since it is defected it occurs once more in a column numbered  $1 \pmod{m}$ .

To complete the proof we show that the first defected integer occurs 3 times. Suppose that  $i_1$  occurs only once in  $T_{\underline{\mu}}$ . Then it occurs strictly more than  $m$  times in the top or bottom row of  $T_{\underline{\tau}}$ . Without loss of generality it occurs strictly more than  $m$  times in the top row of  $T_{\underline{\tau}}$ , and say it occurs in column  $am + 1$ . Suppose  $i_1$  makes its first appearance in  $T_{\underline{\tau}}$  in column  $(a - 1)m + 1 + j$  for  $0 < j \leq m - 1$ . Since it occurs only once in a column numbered  $1 \pmod{m}$ , the total number of occurrences of  $i_1$  in the top row of  $T_{\underline{\tau}}$  is at most  $2m - j$ . So it occurs in the bottom row of  $T_{\underline{\tau}}$  as well. Now suppose its first occurrence in the bottom row of  $T_{\underline{\tau}}$  is in column  $bm + 1 + k$ , for  $0 < k \leq m - 1$ . Since all integers less than  $i_1$  occur  $2m$  times in  $T_{\underline{\tau}}$  it follows that  $j + k = 0 \pmod{m}$ . But  $j + k < 2m$  and so  $j + k = m$ . Now each integer less than  $i_1$  is not defected and so appears twice in  $T_{\underline{\tau}}$  in columns numbered  $1 \pmod{m}$ . The number of such columns available is  $a - 1 + b$  and since this has to be even,  $a + b$  must be odd. But then the total number of positions available for integers less than  $i_1$  in  $T_{\underline{\tau}}$  is  $(a - 1 + b)m + j + k$  which is  $(a + b)m$ , an odd multiple of  $m$ . But each integer less than  $i_1$  appears  $2m$  times in  $T_{\underline{\tau}}$  so, together, they occupy an even number of positions, a contradiction.  $\square$

**Lemma 6.6.** *If  $j$  is odd,  $i_j$  appears in the top row and in the bottom row of  $T_{\underline{\mu}}$ .*

*Proof.* From Lemma 6.5  $i_j$  appears three times in  $T_{\underline{\mu}}$ . If all of them appear consecutively in  $T_{\underline{\mu}}$  then the number of  $i_j$  in  $T_{\underline{\tau}}$  would be greater than  $2m$ , a contradiction.  $\square$

**Notation 6.7.** *Let  $T_{\underline{\tau}}^k$  be the subtableau of  $T_{\underline{\tau}}$  having  $m$  columns starting with  $C_{(k-1)m+1}$  and ending with  $C_{km}$ . We denote the first column of  $T_{\underline{\tau}}^k$  by  $T_{\underline{\tau}}^k[1]$  and the last column as  $T_{\underline{\tau}}^k[m]$ .*

*For  $j$  odd, define  $l(j)$  to be that  $k$  for which  $(k - 1)m + 1 \leq l_{i_j} \leq km$ . So  $T_{\underline{\tau}}^{l(j)}$  is the subtableau containing the last occurrence of  $i_j$  in the bottom i.e containing  $C_{l_{i_j}}$  as one of its  $m$  columns. For  $j$  even let  $f(j)$  be that  $k$  for which  $(k - 1)m + 1 \leq f_{i_j} \leq km$ . So  $T_{\underline{\tau}}^{f(j)}$  is the subtableau containing the first occurrence of  $i_j$  in the bottom row i.e containing  $C_{f_{i_j}}$  as one of its  $m$  columns.*

*For  $j$  odd, let  $S_{\underline{\tau},j}$  denote the subtableau with columns  $T_{\underline{\tau}}^{l(j)}[1], T_{\underline{\tau}}^{l(j)}[m], T_{\underline{\tau}}^{l(j)+1}[1], T_{\underline{\tau}}^{l(j)+1}[m], \dots, T_{\underline{\tau}}^{f(j+1)}[1]C_{f_{i_{j+1}}}$ . Note that this tableau contains an even number of columns since  $T_{\underline{\tau}}^{f(j+1)}[1]$  is different from  $C_{f_{i_{j+1}}}$  - by definition  $i_{j+1}$  appears only once in  $T_{\underline{\mu}}$  and so its first occurrence cannot be in a column numbered  $1 \pmod{m}$  in  $T_{\underline{\tau}}$  by Observation 6.4.*

*We denote by  $S_{\underline{\tau},j}[k]$  the  $2 \times 2$  subtableau of  $S_{\underline{\tau},j}$  containing columns  $2k - 1$  and  $2k$ . To simplify notation we mostly omit the  $\underline{\tau}$  and just denote this by  $S_j[k]$  when  $\underline{\tau}$  is clear from the context. Let  $S_{\underline{\tau},j}[k] =$*

$$S_{\underline{\tau},j}[k] = \begin{array}{|c|c|} \hline p & q \\ \hline r & s \\ \hline \end{array}.$$

*We set  $S_{\underline{\tau},j}[k](1) = p$ ,  $S_{\underline{\tau},j}[k](2) = q$ ,  $S_{\underline{\tau},j}[k](3) = r$  and  $S_{\underline{\tau},j}[k](4) = s$ .*



**Example 6.8.** We illustrate the above notation using the monomial in  $H^0(G_{2,13}, \mathcal{L}(13\omega_2)^{\otimes 2})^T$  given by the following semistandard tableau  $T_\tau$ .

1	1	1	1	2	2	2	2	3	3	3	3	4	7	7	8	8	9	9	9	10	10	10	11	11	12
4	4	4	5	5	5	5	6	6	6	6	7	7	8	8	9	10	11	11	12	12	12	13	13	13	13

For this tableau we have,

$$T_\mu = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 & 3 & 4 & 7 & 8 & 9 & 10 & 10 & 11 \\ \hline 4 & 4 & 5 & 5 & 6 & 6 & 7 & 8 & 10 & 11 & 12 & 13 & 13 \\ \hline \end{array},$$

$$T_\nu = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 & 3 & 7 & 8 & 9 & 9 & 10 & 11 & 12 \\ \hline 4 & 5 & 5 & 6 & 6 & 7 & 8 & 9 & 11 & 12 & 12 & 13 & 13 \\ \hline \end{array}.$$

$\mathcal{D} = \{4, 9, 10, 12\}$  and  $l_4 = 3, f_9 = 16, l_{10} = 17, f_{12} = 20$ . We have  $l(1) = 2, f(2) = 8, l(3) = 9, f(4) = 10$ . Furthermore,

$$S_{\tau,1} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 7 & 7 & 8 \\ \hline 4 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 7 & 7 & 8 & 8 & 9 \\ \hline \end{array}, \quad S_{\tau,2} = \begin{array}{|c|c|c|c|} \hline 8 & 9 & 9 & 9 \\ \hline 10 & 11 & 11 & 12 \\ \hline \end{array}.$$

For the above example

$$S_{\tau,1}[1] = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 4 & 5 \\ \hline \end{array}, \quad S_{\tau,1}[2] = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 5 & 5 \\ \hline \end{array}, \quad S_{\tau,1}[3] = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 5 & 6 \\ \hline \end{array}, \quad S_{\tau,1}[4] = \begin{array}{|c|c|} \hline 3 & 3 \\ \hline 6 & 6 \\ \hline \end{array},$$

$$S_{\tau,1}[5] = \begin{array}{|c|c|} \hline 3 & 3 \\ \hline 6 & 7 \\ \hline \end{array}, \quad S_{\tau,1}[6] = \begin{array}{|c|c|} \hline 4 & 7 \\ \hline 7 & 8 \\ \hline \end{array}, \quad S_{\tau,1}[7] = \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 8 & 9 \\ \hline \end{array},$$

$$S_{\tau,2}[1] = \begin{array}{|c|c|} \hline 8 & 9 \\ \hline 10 & 11 \\ \hline \end{array}, \quad S_{\tau,2}[2] = \begin{array}{|c|c|} \hline 9 & 9 \\ \hline 11 & 12 \\ \hline \end{array}.$$

We will use the degree lexicographic order on rectangular  $2 \times m$  Young tableau. Recall that as per this order a monomial  $p = p_{\tau_1} \dots p_{\tau_m}$  corresponding to a  $2 \times m$  tableau is bigger than a monomial  $q = q_{\mu_1} \dots q_{\mu_{m'}}$  corresponding to another  $2 \times m'$  tableau if  $m > m'$  or, if  $m = m'$ , then for the smallest  $i$  such that  $\tau_i \neq \mu_i$  it is the case that  $\tau_i > \mu_i$  in the usual lexicographic order on words of length 2.

Now we fix a  $j$  which is odd and look at the subtableau  $S_j$  defined above for this  $j$ . Suppose  $S_j$  has  $2t$  columns.

**Lemma 6.9.** For  $1 \leq k < t$  we have  $S_j[k](4) = S_j[k+1](3)$ . If there exists a  $k$  such that  $i_j \leq S_j[k](1) < i_{j+1}$  then  $S_j[k](2) = S_j[k+1](1)$ .

*Proof.* We prove the first statement by contradiction. Suppose  $S_j[k](4) \neq S_j[k+1](3)$  for some  $k$ . Then  $f_{S_j[k+1](3)} \equiv 1 \pmod{m}$ . So we have  $f^{S_j[k+1](3)} \equiv 1 \pmod{m}$  from Observation 6.4. If the number of times  $S_j[k+1](3)$  appears in row 1 or row 2 is not  $m$  then  $S_j[k+1](3)$  would occur 3 times in  $T_\mu$ , a contradiction to the fact that  $S_j[k+1](3)$  is not defectured. So

$S_j[k+1](3)$  appears  $m$  times in row 1 and  $m$  times in row 2. As we iterate over  $k$ , this is true of all the  $S_j[k+1](3)$ 's - they appear  $m$  times in the bottom row starting with a column numbered  $1 \pmod{m}$  and also  $m$  times in the top row starting with a column numbered  $1 \pmod{m}$ . When  $k = f(j+1) - 1$  this means  $f_{i_{j+1}} \equiv 1 \pmod{m}$  and, as argued above,  $f^{i_{j+1}} \equiv 1 \pmod{m}$ . Since  $i_{j+1}$  is defected it has to occur 3 times which is a contradiction to lemma 6.5, since  $j$  is odd.

The proof of the second statement is similar and is omitted.  $\square$

**Lemma 6.10.** *Let  $j$  be odd and suppose that  $S_j$  has  $2t_j$  columns for some  $t_j$ . Then for  $1 \leq k \leq t_j$  we have  $S_j[k](3) > S_j[k](2)$ .*

*Proof.* We prove the lemma for  $j = 1$ . We first show this for  $k = 1$ . Assume  $S_1[1](3) < S_1[1](2)$ . Consider the tableau  $T_\mu$ . The column  $[S_1[1](1), S_1[1](3)]$  is a column numbered  $1 \pmod{m}$  in  $T_\tau$  and so this column appears in  $T_\mu$ . If  $S_1[1](3) < S_1[1](2)$ , then all occurrences of  $S_1[1](3)$  in  $T_\mu$  appear in this column and to the left. The total number of positions in the boxes to the left of this column (including this column) in  $T_\mu$  is an even number. But  $S_1[1](3)$  appears 3 times in these boxes since  $i_1$  has defect 3, and each other integer appears an even number of times since they are not defected. This is a contradiction.

Now we show this for  $k > 1$ . Note that the column  $[S_1[k](1), S_1[k](3)]$  occurs in  $T_\mu$  since it is a column numbered  $1 \pmod{m}$  in  $T_\tau$ . If  $S_1[k](3) > S_1[k](2)$  then all occurrences of  $S_1[k](3)$  in  $T_\mu$  are in this column and to its left. This is true for  $S_1[k](1)$  too. Since  $S_1[k](1)$  and  $S_1[k](3)$  are not defected, they appear twice. The total number of positions to the left of (and including this) column  $[S_1[k](1), S_1[k](3)]$  in  $T_\mu$  is even. This is a contradiction since  $i_1$  appears 3 times and all the other numbers appear twice.

For  $j$  odd and bigger than 1, the proof is similar. Recall that the first column of  $S_{\tau,j}$  is column  $T_\tau^{l(j)}[1]$  and this appears in  $T_\mu$ . The only point to note is that in  $T_\mu$ , the columns strictly to left of the column  $T_\tau^{l(j)}[1]$  contain all occurrences of the previous  $i_k$ ,  $k < j$  and the sum of the occurrences of these  $i_k$ ,  $k < j$  is even. So too is the sum of occurrences of the remaining integers since they are not defected. The argument then proceeds as in the  $j = 1$  case.  $\square$

**Proposition 6.11.** *The map  $R(1)^{\otimes m} \rightarrow R(m)$  is surjective.*

*Proof.* The proof is by induction. For  $p_\tau$  in  $R(m)$ , we show that there exists  $p_\mu \in R(1)$  and  $p_\nu \in R(m-1)$  and  $p_{\tau^j} \in R(m)$  such that  $p_\tau = p_\nu p_\mu + \sum_j p_{\tau^j}$  with  $p_{\tau^j} < p_\tau$  in lexicographic order. An induction based on degree lexicographic order on monomials completes the proof.

The base case - the lexicographically smallest monomial  $p_\tau$  in  $R(m)$ , is the one corresponding to the semistandard Young tableau filled with  $[1, 2]$  in the first  $2m$  columns and then  $[3, 4]$  and so on. Fix  $0 \leq j \leq m-1$ . Consider the monomial  $p_{\tau^j}$  associated to the subtableau of this tableau consisting of columns  $1+j, m+1+j, 2m+1+j, \dots, (n-1)m+1+j$ . This is a zero weight vector in  $R(1)$ . The product of  $p_{\tau^j}$  is the lexicographically smallest monomial  $p_\tau$  in  $R(m)$ .

In general starting with  $p_{\underline{\tau}}$  we construct  $p_{\underline{\mu}}$  and  $p_{\underline{\nu}}$  as given before Definition 6.1, by taking for  $T_{\underline{\mu}}$  the subtableau with columns  $1, m+1, \dots, (n-1)m+1$ . If  $p_{\underline{\mu}}$  is a zero weight vector (i.e in the corresponding tableau no integer is defected) we are done. The monomial  $p_{\underline{\tau}}$  is the product of a zero weight vector in  $R(1)$  and an element in  $R(m-1)$  and we are done by induction on degree.

Otherwise, proceeding as above we have defected integers  $\{i_1, i_2, \dots, i_{2l}\}$ . Corresponding to the integer  $i_j$  in  $\{i_1, i_3, \dots, i_{2l-1}\}$ , we have subtableaux  $S_j$  and Lemma 6.10 holds. For  $j$  odd let the number of columns in  $S_j$  be  $2t_j$ .

Case 1 : Suppose for all  $j$  and  $1 \leq k \leq t_j$  it is the case that  $S_j[k](3) > S_j[k](2)$ .

In this case we do the following operation : We swap  $S_j[k](3)$  and  $S_j[k](4)$  and keep  $S_j[k](1), S_j[k](2)$  fixed for all  $j$  odd and for all  $1 \leq k \leq t_j$ . We get a new Young tableau, call it  $S'_j$ . We modify the original tableau corresponding to  $p_{\underline{\tau}}$  by replacing the columns which were previously used to get  $S_j$  by the corresponding columns of  $S'_j$ . We do this for every  $j$ .

Denote the new monomial computed by this tableau by  $p_{\underline{\tau}'}$  and denote by  $p_{\underline{\mu}'}$  the monomial obtained from this tableau by selecting columns numbered  $1, m+1, 2m+1, \dots, (n-1)m+1$ . It is clear that  $T_{\underline{\mu}'}$  is semistandard. Furthermore for every  $j$  odd, one of the  $i_j$ 's which appeared in a column numbered  $1 \pmod{m}$  in  $T_{\underline{\mu}}$  appears now in a column numbered  $0 \pmod{m}$ , and so its count in  $T_{\underline{\mu}'}$  is one less than in  $T_{\underline{\mu}}$ . So  $i_j$  is not defected in  $T_{\underline{\mu}'}$ . For this same  $j$  the last exchange is done between  $S_j[t_j](3)$  and  $S_j[t_j](4)$  and this is  $i_{j+1}$ . So this  $i_{j+1}$  now occurs in a column numbered  $1 \pmod{m}$  in  $T_{\underline{\tau}'}$ , and so the count of  $i_{j+1}$  in  $T_{\underline{\mu}'}$  is one more than in  $T_{\underline{\mu}}$ . So  $i_{j+1}$  is not defected in  $T_{\underline{\mu}'}$ . From Lemma 6.9,  $S_j[k](4) = S_j[k+1](3)$  for all  $k$ , so the counts of the remaining integers in  $T_{\underline{\mu}'}$  are the same as their count in  $T_{\underline{\mu}}$ , so these continue to be not defected. This is true for every  $j$ . So no integer is defected in  $T_{\underline{\mu}'}$  and the corresponding monomial is a zero weight vector in  $R(1)$ . So  $p_{\underline{\tau}'}$  is a product of a zero weight monomial in  $R(1)$  and an element of  $R(m-1)$ .

To finish the proof in this case we compare  $p_{\underline{\tau}}$  with  $p_{\underline{\tau}'}$ . Let us denote the set of columns of  $T_{\underline{\tau}}$  not in any  $S_j$  by  $Q$  and the monomial computed by them as  $y$ . If  $S_j$  has  $2t_j$  columns the monomial computed by it is a product of the  $t_j$  monomials computed by the  $2 \times 2$  subtableaux  $S_j[k]$ ,  $p_{S_j[k]} := p_{(S_j[k](1), S_j[k](3))} p_{(S_j[k](2), S_j[k](4))}$ . We have

$$p_{\underline{\tau}} = y \prod_{j=1}^{j=l} \prod_{k=1}^{k=t_j} p_{S_j[k]}, \quad (3)$$

$$p_{\underline{\tau}'} = y \prod_{j=1}^{j=l} \prod_{k=1}^{k=t_j} p_{S'_j[k]}. \quad (4)$$

From the straightening laws the following relation holds between the tableaux  $S_j[k]$  and  $S'_j[k]$ .

$$\begin{array}{|c|c|} \hline p & q \\ \hline r & s \\ \hline \end{array} = \begin{array}{|c|c|} \hline p & q \\ \hline s & r \\ \hline \end{array} \pm \begin{array}{|c|c|} \hline p & r \\ \hline q & s \\ \hline \end{array} \quad (5)$$

Recall that in the equation above  $S_j[k]$  is the tableau on the left hand side of the equation and  $S'_j[k]$  is the first tableau on the right hand side.

Plugging this into Equation 3 above we see that  $p_{\underline{\tau}}$  is the sum of  $p_{\underline{\tau}'}$  and sums of products of monomials obtained from  $p_{\underline{\tau}}$  by replacing at least one of the terms  $p_{S_j[k]}$  in its expression by  $p'_j[k]$ , the monomial computed by the second tableau on the right hand side of Equation 5. However from the hypothesis of this case  $r > q$ . So the second tableau on the right is lexicographically smaller than the tableau  $S_j[k]$ . So the  $2 \times mn$  tableau corresponding to each additional term obtained by plugging Equation 5 into Equation 3 is lexicographically smaller than  $T_{\underline{\tau}}$ . It is possible that this tableau is not semistandard and needs to be straightened into a sum of semistandard tableaux. But each such tableau  $T_{\underline{\tau}''}$ , will be lexicographically smaller than the (non semistandard) tableau we started with. We proved above that  $p_{\underline{\tau}'}$  is the product of  $p_{\underline{\mu}'} \in R(1)$  and a monomial  $p_{\underline{\nu}'} \in R(m-1)$ . We have

$$p_{\underline{\tau}} = p_{\underline{\mu}'} p_{\underline{\nu}'} + \sum_s p_{\underline{\tau}''_s},$$

with the sum being over tableaux which are smaller than  $\tau$  in lexicographic order. By induction on lexicographic order each of these is in the image of  $R(1)^{\otimes m}$ . By induction on degree  $p_{\underline{\nu}'}$  is in the image of  $R(1)^{\otimes(m-1)}$ . So we are done.

Case 2 : For  $j$  in which the conditions of Case 1 hold we do exactly as in that case. Let  $j$  be such that  $S_j[k](3) = S_j[k](2)$  for some  $1 \leq k < t_j$ . For each such  $j$  we do the following. First note that for such a  $j$ ,  $i_j$  appears in  $S_j$  as  $S_j[k](1)$  for some  $1 < k < t_j$ , since it has defect 3. Let  $z$  be the set of all elements  $i_j \leq z < i_{j+1}$  with  $z = S_j[k](3)$  in some subtableau  $S_j[k]$  with  $S_j[k](3) = S_j[k](2)$ . Order this set as  $\{z_1, z_2, \dots, z_e\}$  such that  $i_j \leq z_1 < z_2 < z_3 \dots < z_e < i_{j+1}$ , and let  $k_i$  denote the index for which  $z_i = S_j[k_i](3) = S_j[k_i](2)$  - clearly  $z_s \neq z_t$  for  $s \neq t$  and  $z_e < i_{j+1}$  and  $k_i \geq k$ . Let  $x_i, y_i$  denote  $S_j[k_i](1)$  and  $S_j[k_i](4)$ . For  $k < k_i < t_j$  it follows from Lemma 6.9 that  $S_j[k_i-1](4) = z_i$ ,  $S_j[k_i+1](1) = z_i$ ,  $S_j[k_i-1](2) = x_i$  and  $S_j[k_i+1](3) = y_i$ . We have two subcases.

- i  $e$  is odd: In this case we first swap  $S_j[l](1)$  and  $S_j[l](2)$  for all  $k \leq l \leq k_1 - 1$ . Then swap the two columns in  $S_{k_1}$ . And swap  $S_j[l](3)$  and  $S_j[l](4)$  for all  $k_1 + 1 \leq l \leq k_2 - 1$ . Do nothing with  $S_j[k_2]$ . Instead start with  $z_2$  which appears in  $S_j[k_2+1](1)$  and  $S_j[k_2+1](2)$  and repeat these steps. Since  $e$  is odd, the last set of swaps will happen in the bottom row starting from  $y_e = S_j[k_e+1](3)$  up to  $i_{j+1} = S_j[t_j](4)$ .
- ii  $e$  is even: In this case we swap  $S_j[l](3)$  and  $S_j[l](4)$  for all  $1 \leq l \leq k_1 - 1$ . Do nothing with  $S_j[k_1]$ . Instead swap  $S_j[l](1)$  and  $S_j[l](2)$  for all  $k_1 + 1 \leq l \leq k_2 - 1$  and then swap the two columns of  $S_j[k_2]$ . And repeat the procedure from the  $y_2$  which appears as  $S_{k_2+1}[l](3)$ . Since  $e$  is even it can be checked that the last swaps will happen in the bottom row from  $y_e = S_j[k_e+1](3)$  to  $i_{j+1} = S_j[t_j](4)$ .

After these round of swaps, we can use straightening as we did in case 1 above, to complete the proof. The last set of swaps take place in the bottom row starting with an element occurring in a  $1 \pmod{m}$  position and ending with the first occurrence of  $i_{j+1}$  in the bottom row - this is true in both cases. In both cases the first set of swaps start

with  $i_j$  occurring in a  $1 \pmod{m}$  position and end with an element occurring in a position  $0 \pmod{m}$ . It can be checked that if we form tableau  $S_{j'}$  as we did in Case 1 above, the number of  $i_j$  has reduced and the number of  $i_{j+1}$  has increased. The number of occurrences of the intermediate numbers does not change because of the column swaps performed. Furthermore, the other set of swaps between elements in the top row and elements in the bottom row in an  $S_j[l]$  take place in those  $l$  wherein  $S_j[l](3) > S_j[l](2)$ . One checks as in Case 1 above that straightening introduces new zero weight tableaux, but all of them are lexicographically smaller than the tableau we start with. This completes the proof.  $\square$

**Example 6.12.** We continue with the example in 6.8. We reproduce  $T_\tau$  below. For the subtableau  $S_{\tau,1}$  of  $T_\tau$  we perform the operations suggested in Case 2 above, and for the subtableau  $S_{\tau,2}$  we perform the operations as in Case 1. So we can write the monomial given by

1	1	1	1	2	2	2	2	3	3	3	3	4	7	7	8	8	9	9	9	10	10	10	11	11	12
4	4	4	5	5	5	5	6	6	6	6	7	7	8	8	9	10	11	11	12	12	12	13	13	13	13

as a sum of the monomials given by the following two tableaux,

1	1	1	1	2	2	2	2	3	3	3	3	4	7	8	7	8	9	9	9	10	10	10	11	11	12
4	4	5	4	5	5	6	5	6	6	7	6	7	8	9	8	11	10	12	11	12	12	13	13	13	13

1	1	1	1	2	2	2	2	3	3	3	3	4	7	7	8	8	10	9	9	10	10	10	11	11	12
4	4	4	5	5	5	5	6	6	6	6	7	7	8	8	9	9	11	11	12	12	12	13	13	13	13

The first term of the sum can be written as a product of two zero weight vectors. The odd numbered columns give rise to the zero weight vector represented by the tableau

1	1	2	2	3	3	4	8	8	9	10	10	11
4	5	5	6	6	7	7	9	11	12	12	13	13

The even numbered columns give rise to the zero weight vector represented by the tableau

1	1	2	2	3	3	7	7	9	9	10	11	12
4	4	5	5	6	6	8	8	10	11	12	13	13

The tableau corresponding to the second summand is nonstandard but lexicographically smaller than  $T_\tau$ .

**Theorem 6.13.**  $(T \setminus (G_{2,n}))_T^{ss}(\mathcal{L}(n\omega_2), \tilde{\mathcal{L}}(n\omega_2))$  is projectively normal.

*Proof.* Now

$$(T \setminus (G_{2,n}))_T^{ss}(\mathcal{L}(n\omega_2), \tilde{\mathcal{L}}(n\omega_2))$$

is normal.  $R_1$ -generation follows from Proposition 6. The theorem follows.  $\square$

**Corollary 6.14.** *The GIT quotient of a Schubert variety in  $G_{2,n}$  is projectively normal with respect to the descent line bundle.*

*Proof.*  $T$  is a linearly reductive group. For a Schubert variety  $X(w)$  in  $G(2, n)$  the map  $H^0(G_{2,n}, \mathcal{L}(n\omega_2)^{\otimes m})^T \rightarrow H^0(X(w), \mathcal{L}(n\omega_2)^{\otimes m})^T$  is surjective. Since  $X(w)_{T}^{ss}(\mathcal{L}(n\omega_2))$  is normal the corollary follows.  $\square$

We have an analogue of Corollary 5.8. The proof is similar and is omitted.

**Corollary 6.15.** *The GIT quotient of a Richardson variety in  $G_{2,n}$  is projectively normal with respect to the descent line bundle.*

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