A GEOMETRIC APPROACH TO THE KRONECKER PROBLEM II: INVARIANTS OF MATRICES FOR SIMULTANEOUS LEFT-RIGHT ACTIONS

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Abstract. In this paper we describe the ring of invariants of the space of m-tuples of $n \times n$ matrices, under the action of $\mathrm{SL}(n) \times \mathrm{SL}(n)$ given by $(A, B) \cdot (X_1, X_2, \cdots, X_m) \mapsto (AX_1B^t, AX_2B^t, \cdots, AX_mB^t)$. Determining the ring of invariants is the first step in the geometric approach to finding multiplicities of representations of the symmetric group in the tensor product of rectangular shaped representations. We show that these invariants are given by *multi-determinants* and can also be described in terms of certain magic squares. We compute the null cone for this action. We also study a birational subring of invariants and an analysis thereof results into a different proof of the Artin-Procesi theorem for the ring of invariants for several matrices under the simultaneous conjugation action of $\mathrm{SL}(n)$.
Keywords: Invariant theory; Artin-Procesi theorem; Null cone

1. Introduction

The basic problem addressed in this paper concerns describing the ring of invariants, over say the field of complex numbers $\mathbb{C}$, of m-tuples of $n \times n$ matrices under the left-right action by the direct product $\mathrm{SL}(n) \times \mathrm{SL}(n)$ of two copies of the special linear group. This problem naturally arose out of the earlier work of the first and third author in [1]. Let us briefly recall the underlying motivations.

Let S_k denote the symmetric group on k letters. It is well known (see [6]) that the complex irreducible representations of S_k are parameterized by integer partitions of k. As we are primarily interested in complex representations, by 'representation' we always mean a complex representation. Let λ, μ and ν be three partitions of the integer k. Further, let W_λ, W_μ and W_ν be the associated irreducible representations of S_k. Under the natural diagonal action of S_k, $W_\mu \otimes W_\nu$ becomes a representation of S_k. The Kronecker problem, in this context, is to 'compute' the multiplicity $m_{\lambda\mu\nu}$ with which the representation W_λ occurs in the representation $W_\mu \otimes W_\nu$.

Our interest in this problem is motivated by the work 'Geometric Complexity Theory' of Mulmuley and Sohoni, [9, 10]. In this work, a strong link is established between the separation of complexity classes in Computer Science and algorithmic problems in Representation Theory. It has been shown there that a good understanding of the 'subgroup restriction problem' (see [10]) will be an important step in demonstrating separation of complexity classes via the proposed approach. The Kronecker problem, of determining tensor product multiplicities of the symmetric group is a special, albeit very important case of the 'subgroup restriction problem' [10]. However very little is known about this problem. The problem has been solved when λ and μ are partitions with at most two parts [14], and when they are both hook shaped [15].

In [1] a geometric approach was proposed to study this problem. The approach was based on the well known Schur-Weyl duality [16, 7]. Assume that λ has atmost m rows, μ has atmost n rows and ν has atmost p rows. Then λ parameterizes an irreducible representation, V_λ, of $\mathrm{GL}(m)$, μ parameterizes an irreducible representation, V_μ, of $\mathrm{GL}(n)$ and ν parameterizes a representation, V_ν, of $\mathrm{GL}(p)$ (see [6]). Now we have a natural action of $\mathrm{GL}(m) \times \mathrm{GL}(n) \times \mathrm{GL}(p)$ on $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^p$, and so an induced action on $Sym^k(\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^p)$. Note that the k here is the same as the parameter in the symmetric group S_k whose tensor product multiplicities we wish to determine. In [1] it was shown that $m_{\lambda\mu\nu}$ is equal to the multiplicity of the $\mathrm{GL}(m) \times \mathrm{GL}(n) \times \mathrm{GL}(p)$ module $V_\lambda \otimes V_\mu \otimes V_\nu$ in $Sym^k(\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^p)$. In [1] this problem was completely solved in the case when $m = n = p = 2$ using geometric methods.

In this paper we address the following question - determine $m_{\lambda\mu\nu}$ in the case when μ and ν are rectangular shapes. We make a further assumption that μ and ν are identical with exactly $n = p$ rows. Under these assumptions, V_μ and V_ν are both $\mathrm{GL}(n)$ modules. This viewpoint of the problem brings in a lot of geometry and seems promising. This is because $\mathrm{GL}(n)$ irreducible modules parameterized by rectangular shapes with n rows are precisely the semi-invariants for the given action.

So, the problem is to study $\mathrm{GL}(m)\times\mathrm{GL}(n)\times\mathrm{GL}(n)$ acting on $Sym^k(\mathbb{C}^m\otimes\mathbb{C}^n\otimes\mathbb{C}^n)$, and understand the $\mathrm{GL}(m)$ module structure of the $\mathrm{GL}(n) \times \mathrm{GL}(n)$ semi-invariants. In other words, working instead with $\mathrm{SL}(m) \times \mathrm{SL}(n) \times \mathrm{SL}(n)$, we ask:

(A) What are the $\mathrm{SL}(n) \times \mathrm{SL}(n)$ invariants of $Sym^k(\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^n)$?
(B) How does the invariant space in (A) decompose as an $\mathrm{SL}(m)$ module? We are interested, of course, in an explicit decomposition rule.

We give a complete solution to question (A) above. We also get a combinatorial model for the invariant space, which we believe will be useful to solve question (B) above.

Now identify $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ with m-tuples of $n \times n$ matrices $X = (X_1, X_2, \cdots, X_m)$. Let $\mathrm{SL}(n) \times \mathrm{SL}(n)$ act on X as,

$$(A, B) \circ (X_1, X_2, \cdots, X_m) = (AX_1B^t, AX_2B^t, \cdots, AX_mB^t).$$

Thinking of X_i as the i-th column of an $n^2 \times m$ matrix, we get a natural action of $\mathrm{SL}(m)$ on X which commutes with the action of $\mathrm{SL}(n) \times \mathrm{SL}(n)$ just defined. It can be easily checked that the total action of $\mathrm{SL}(m) \times \mathrm{SL}(n) \times \mathrm{SL}(n)$ on X, is compatible with the natural action of $\mathrm{SL}(m) \times \mathrm{SL}(n) \times \mathrm{SL}(n)$ on $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^n$.

And so the answer to question (A) above is - these invariants are precisely the degree k invariants of m-tuples of $n \times n$ matrices under the natural action of $\mathrm{SL}(n) \times \mathrm{SL}(n)$. So what we need to know is a description of this ring of invariants $R_{m,n}$ and the $\mathrm{SL}(m)$-structure of the k-th graded piece of this ring.

Now we recall a closely related problem which is well studied in the literature. Consider the action of $\mathrm{SL}(n)$ on l-tuples of $n \times n$ matrices, $(Y_1, Y_2, \ldots, Y_l)$ by simultaneous conjugation

$$C \circ (Y_1, Y_2, \cdots, Y_l) = (CY_1C^{-1}, CY_2C^{-1}, \cdots, CY_lC^{-1}).$$

For this action, in characteristic 0, Artin [2] conjectured that every invariant is a polynomial in the elements $\mathrm{Tr}(Y_{i_1}Y_{i_2}\cdots Y_{i_r})$; traces of monomials in $Y_i's$. In [13], Procesi proved this conjecture. He also showed that the ring of invariants $P_{l,n}$ (for the conjugation action) is generated by trace elements of the form $\mathrm{Tr}(Y_{i_1}Y_{i_2}\cdots Y_{i_r})$, with $r \leq 2^n - 1$.

Returning to our left-right action of $\mathrm{SL}(n) \times \mathrm{SL}(n)$ on $X = (X_1, X_2, \cdots, X_m)$, let $Y_{ij} = X_i X_j'$ and $Z_{ij} = X_i' X_j$, where for a matrix M, M' denotes its adjoint. Then $(A, B) \circ Y_{ij} = AY_{ij}A^{-1}$ and $(A, B) \circ Z_{ij} = (B^t)^{-1} Z_{ij} B^t$. Therefore the trace of any monomial in Y_{ij} and Z_{ij} is an invariant for our action. In view of the Artin-Procesi theorem, one might expect that the ring of invariants $R_{m,n}$ for the simulataneous left-right action equals the subring generated by these trace elements. Indeed, this turns out to be the case when $n = 2$. However for bigger n, the statement is not true. What is true, however, is that this subring of invariants is birational to the full ring of invariants. This observation coupled with our new description of the ring $R_{m,n}$ allows us to give another proof of the Artin-Procesi theorem - albeit not as elegant as Procesi's. We should point out here that our theorem appears to be more general - while we can derive Procesi's theorem from ours, it is not clear how to derive our results from Procesi's theorem.

Naturally our new description of $R_{m,n}$ is inspired by the beautiful paper of Procesi. The technique of proof is also similar in spirit to Procesi's, and, in Procesi's language, 'quite simple'. As in Procesi's work, it suffices to find multilinear invariants (cf. ([4], Chapter I) and [16]). We first observe that multilinear invariants of degree k exist only when k is a multiple of n. We then give a matrix-theoretic description of each multilinear invariant. We show that each multilinear invariant of degree nd is a specialization of the *multi-determinant* of degree nd. By the multideterminant of degree nd, we mean the complete polarization of the determinant of a $nd \times nd$ matrix of indeterminates.

We then give a combinatorial description of the multilinear invariants - we show that for all d, every multilinear invariant of degree nd is parameterized by a $d \times d$ matrix with row and column sum equal to n. For want of a better term, we call such matrices magic squares.

This allows us to succinctly describe a generating set for the invariants of degree nd - they are the coefficients of the monomials which occur in the determinant of a generic $nd \times nd$ matrix (see Theorem 11 for an exact statement). This theorem also allows us to extract a combinatorial model for the invariant space (see Remark 12). We believe that this combinatorial model will be useful in determining the $\mathrm{SL}(m)$ structure of the ring of invariants, thus settling question (B) above. This in ongoing work.

Our solution to question (A) above may also be seen in the light of classical invariant theory. The first fundamental theorem of invariant theory settles the question of invariants, when $\mathrm{GL}(n)$ acts diagonally on an arbitrary number of copies of its defining representation space $\mathbb{C}^n$. Procesi's theorem settles question of invariants, when $\mathrm{GL}(n)$ acts diagonally on an arbitrary number of copies of its adjoint representation space. Our result settles the question of invariants, when $\mathrm{SL}(n) \times \mathrm{SL}(n)$ acts diagonally on an arbitrary number of copies of its defining representation space, $\mathbb{C}^n \otimes \mathbb{C}^n$.

The outline of our paper is as follows. In section 2 we develop the necessary notation and recall some results from classical invariant theory. We describe the multilinear invariants for our action. In the section 3 we show that every multilinear invariant is in fact a specialization of the multi-determinant. We then describe how each multilinear invariant is parameterized by a magic square. We then use this to give a succinct description of the generators of a homogeneous piece of the ring of invariants. We also describe the invariants combinatorially. In section 4 we show that a localization of $R_{m,n}$, is closely related to $P_{m-1,n}$. We deduce Procesi's result as a consequence. In section 5 we restrict ourselves to the $m = 2$ case. We show that the ring of invariants is a polynomial ring and exhibit generators for the same. We then consider the $n = 2$ case, and give a set of generators for the ring of invariants. In section 6 we give a description of the null cone. We conclude in section 7 with some open problems and ongoing work.

Remark. The results in sections 5, 6 already appear in [3]. We discovered these results independently, and stumbled upon the above paper, quite by accident, just as we were writing the paper for submission. We have still included these results for the sake of completeness. In [3], the authors consider the problem of computing the null-cone for the left-right action of $G = \mathrm{SL}(n) \times \mathrm{SL}(n)$ on m-tuples of matrices (a problem we address in section 6). The authors then show that a complete polarization of the determinant function determines the null-cone in the case when $m = 2$ or when $n = 2$ (see Theorem 2.7 in [ibid]). This is precisely the content of our theorems 22, 24 given in section 5 below. Our proofs are however different and straightforward.

2. Multilinear invariants for $n \times n$ matrices.

Before describing the main problem, let us fix some notation. Let K be an algebraically closed field of characteristic zero; K^n denotes the n-dimensional vector space. The general linear group is denoted by $\mathrm{GL}(n, K)$ or simply $\mathrm{GL}(n)$ while $\mathrm{SL}(n, K)$ or $\mathrm{SL}(n)$ denotes the subgroup of matrices of determinant 1. We denote the ring of $n \times n$ matrices by $\mathrm{M}(n, K)$ or simply $\mathrm{M}(n)$ or M_n. The k-fold symmetric power of a vector space V shall be denoted by $\mathrm{S}^k(V)$.

Set $G := \mathrm{SL}(n) \times \mathrm{SL}(n)$. Set $\mathcal{X}^{(m)} := \mathrm{M}_n^{\oplus m}$ the space of m-tuples of $n \times n$ matrices. The group G acts rationally on $\mathcal{X} = \mathcal{X}^{(m)}$ according to the rule

$$(A, B) \cdot (X_1, X_2, \ldots, X_m) = (AX_1B^t, AX_2B^t, \ldots, AX_mB^t) \tag{2.1}$$

where $(A, B) \in G$ and $(X_1, X_2, \ldots, X_m) \in \mathcal{X}$.

Our aim is to describe the ring $R_{m,n}$ of G-invariant polynomial functions on $\mathcal{X}$. Clearly $R_{m,n}$ is a graded subring of the coordinate ring of $\mathcal{X}$ under the natural grading that assigns to each monomial, its degree. Therefore it suffices to concentrate on describing the individual graded pieces of $R_{m,n}$.

To narrow down further, one can linearize the problem using classical methods, wherein to any homogeneous invariant (of degree k) one associates a multilinear invariant (on k copies of the original space) via polarization and working backwards, one recovers invariants from multilinear invariants via substitutions, (see [4], Chapter I). In abstract terms, if the field K has characteristic zero, then for any group G acting on a finite-dimensional vector space U and for any integer k, the polarization process produces an inclusion $\mathsf{p}\colon (\mathrm{S}^k(U^*))^G \hookrightarrow ((U^{\otimes k})^*)^G$ while substitution corresponds to a map $\mathsf{s}\colon ((U^{\otimes k})^*)^G \to (\mathrm{S}^k(U^*))^G$ and sp is identity on $(\mathrm{S}^k(U^*))^G$. Let us identify $(U^{\otimes k})^* = (U^*)^{\otimes k}$. One obtains p for instance, by applying $(-)^G$ to a splitting of the natural G-surjection $(U^*)^{\otimes k} \to \mathrm{S}^k(U^*)$, the splitting being given by $f_1 \cdots f_k \mapsto (1/k!) \sum_{\sigma \in S_k} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)}$ where f_i's are functionals on U. For s, we apply $(-)^G$ to the sequence of G-maps $(U^*)^{\otimes k} \hookrightarrow \mathrm{S}^k((U^*)^{\oplus k}) \cong \mathrm{S}^k((U^{\oplus k})^*) \to \mathrm{S}^k(U^*)$, the last map being induced by the diagonal embedding of U in $U^{\oplus k}$.

Keeping the notation and assumptions as in the preceding paragraph, suppose further that U has a decomposition $U = M^{\oplus r}$ as G-modules. Then $(U^*)^{\otimes k}$ admits a natural decomposition into a direct sum of various copies of $(M^*)^{\otimes k}$. In particular, for any invariant in $(\mathrm{S}^k(U^*))^G$, the corresponding multilinear invariant in $((U^{\otimes k})^*)^G$ is a sum of multilinear invariants in $((M^{\otimes k})^*)^G$.

We remark here that what we have called polarization is actually called the complete polarization in classical language. For partial polarization, one introduces for each coordinate function x_i on the space U, a new variable y_i and then applies $\sum_i y_i \partial/\partial x_i$ to a homogeneous invariant. Clearly this too is useful only in characteristic 0. Finally for multihomogeneous polynomials on products of spaces, polarization is achieved by polarizing on each space separately.

Returning to our situation at hand, we may thus reduce the problem of describing the degree k invariant polynomials on $\mathcal{X}^{(m)}$ to that of describing the invariant multilinear functions on k copies of $\mathcal{X}^{(1)} = \mathrm{M}_n$, i.e., elements of $((\mathrm{M}_n^{\otimes k})^*)^G$. Now we shall apply the fundamental theorems of invariant theory for $\mathrm{SL}(n)$.

Let $V = K^n = W$. Note that for our choice of G-action on $\mathrm{M}_n = \mathcal{X}^{(1)}$, there is an isomorphism of G-modules $\mathrm{M}_n \cong V \otimes W$ where the G-action on $V \otimes W$ results via the tensor product of the standard action, namely,

$$(A, B) \cdot (v \otimes w) = Av \otimes Bw.$$

It follows that there are natural isomorphisms

$$((\mathrm{M}_n^{\otimes k})^*)^G \;\cong\; ((V^{\otimes k})^* \otimes (W^{\otimes k})^*)^G \;\cong\; ((V^{\otimes k})^*)^{\mathrm{SL}(n)} \otimes ((W^{\otimes k})^*)^{\mathrm{SL}(n)}.$$

For $1 \le i \le k$, set $V_i = K^n = W_i$. For any k-tuple $\vec{X} = (X_1, \dots X_k)$ of $n \times n$ matrices, we shall think of X_i as an element of $V_i \otimes W_i$. Let $\{e_1, \dots, e_n\}$ denote the standard basis of K^n.

Suppose k is a multiple of n, say $k = nd$. To any permutation σ of $\{1, \dots, k\}$ we associate a functional s_σ on $V_1 \otimes \cdots \otimes V_k$ via the natural surjections

$$\begin{aligned} V_1 \otimes \cdots \otimes V_k &\cong (V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}) \otimes \cdots \otimes (V_{\sigma(k-n+1)} \otimes \cdots \otimes V_{\sigma(k)}) \\ &\to (\wedge^n V) \otimes \cdots \otimes (\wedge^n V) \cong K \otimes \cdots \otimes K \cong K \end{aligned}$$

where the isomorphism $(\wedge^n V) \cong K$ is the one sending $e_1 \wedge \cdots \wedge e_n$ to 1. This functional is $\mathrm{SL}(n)$-invariant for the standard action of $\mathrm{SL}(n)$ on each V_i. Let t_σ denote the $\mathrm{SL}(n)$-invariant functional $W_1 \otimes \cdots \otimes W_k \to K$ obtained by replacing the V_i's with W_i's in the definition of s_σ. Thus for any two permutations σ, τ of $\{1, \dots, k\}$, $s_\sigma \otimes t_\tau$ can be identified as an element of $((\mathrm{M}_n^{\otimes k})^*)^G$.

Recall that by classical invariant theory, $V_1 \otimes \cdots \otimes V_k$ admits a nonzero invariant functional if and only if k is a multiple of n, say $k = dn$, and moreover the space of invariant functionals on $V_1 \otimes \cdots \otimes V_k$ is spanned by $\{s_\sigma \,|\, \sigma \in S_k\}$. It follows that for any such k, $((\mathrm{M}_n^{\otimes k})^*)^G$ is spanned by elements of the kind $s_\sigma \otimes t_\tau$. These are the multilinear invariants which, via substitution, give us the degree k homogeneous invariants on $\mathrm{M}_n^{\oplus m}$ for any m. Of course, substitution does not change the degree of the invariant.

Let us summarize our discussion so far.

Theorem 1. *Let $G = \mathrm{SL}(n) \times \mathrm{SL}(n)$ act on the space $\mathcal{X}^{(m)}$ of m-tuples of $n \times n$ matrices as in* (2.1) *above. Then*

(1) *The degree of any nonzero homogeneous invariant function f on $\mathcal{X}^{(m)}$ is necessarily a multiple of n.*
(2) *Any such invariant f is a K-linear combination of invariants obtained via substitution from multilinear ones of the kind $s_\sigma \otimes t_\tau$ as defined above.*

In the next section we give more matrix-theoretic interpretations of these invariants.

Example 2. Let $n = 2$. Let $\{x_{11}, x_{12}, x_{21}, x_{22}\}$ denote the coordinate functions on the first matrix entry of $\mathcal{X}^{(m)}$. Then

$$(X_1, \dots, X_m) \mapsto \det(X_1) = x_{11}x_{22} - x_{12}x_{21}$$

is a degree 2 invariant on $\mathcal{X}^{(m)}$. Polarizing it gives rise to a multilinear invariant, which, using an additional set of variables, say $\{y_{11}, y_{12}, y_{21}, y_{22}\}$, can be written as

$$x_{11}y_{22} - x_{12}y_{21} + y_{11}x_{22} - y_{12}x_{21}.$$

This is the multideterminant of two matrices (represented by $(x_{ij}), (y_{kl})$). This multilinear invariant is also the same as $s_\sigma \otimes t_\tau$ where σ and τ are the identity permutation in S_2.

In the definition of the multilinear invariant s_σ, the role of $\sigma \in S_{nd}$ is mainly to provide a partition of the set $\{1, \dots, nd\}$ into d ordered parts of n elements each. Therefore it will often be convenient to represent σ via the corresponding partition.

Example 3. Let $n = 3$ and $k = 9$. Let σ be the permutation in S_9 that gives rise to the partition $(1, 3, 6|2, 4, 8|5, 7, 9)$ and let τ be the identity permutation. Then the corresponding invariant $s_\sigma \otimes t_\tau$ on 9 matrices sends $(v_1 \otimes \cdots \otimes v_9) \otimes (w_1 \otimes \cdots \otimes w_9)$ to the element

$$(v_1 \wedge v_3 \wedge v_6)(v_2 \wedge v_4 \wedge v_8)(v_5 \wedge v_7 \wedge v_9) \cdot (w_1 \wedge w_2 \wedge w_3)(w_4 \wedge w_5 \wedge w_6)(w_7 \wedge w_8 \wedge w_9).$$

3. Multideterminants and magic squares

Our aim now is to give appropriate matrix-theoretic descriptions of the invariants in Theorem 1 above. As is to be expected, these descriptions rely on computing suitable determinants and their multilinear avatars called multi-determinants.

To begin with let us recall the notion of a multi-determinant which is nothing but the (complete) polarization of the determinant function. In characteristic-free terms it means the following.

Definition 4. The multideterminant of p matrices $A_1, \dots, A_p$ of size $p \times p$ is the sum of determinants of $p!$ matrices indexed by the elements of the symmetry group S_p wherein to any $\pi \in S_p$ we associate the matrix whose i-th column is the i-th column of $A_{\pi(i)}$. We denote the multideterminant as multi-det$(A_1, \dots, A_p)$.

Since multi-det is multilinear in the matrices A_i involved, we may also think of it as a functional on $\mathrm{M}_p^{\otimes p}$. One can check that the multi-det is in fact the functional $s_\sigma \otimes t_\tau$ defined in §2 where $\sigma = \tau =$ the identity permutation in S_p.

For the purpose of describing the invariant functions on $\mathcal{X}$ of §2, we shall first embed each matrix from our original set into a bigger matrix in the form of a block. This motivates the following definition.

Definition 5. A collection of d^2 matrices $\{N_{ij}\}_{1\le i,j\le d}$ of size $n\times n$ is said to form an $n\times n$-tiling of an $nd\times nd$ matrix M (or M is said to admit a tiling by $\{N_{ij}\}$) if the (p,q)-th entry of N_{ij} equals the $((i-1)n+p,(j-1)n+q)$-th entry of M. In other words, if we imagine M as consisting of d^2 blocks (= tiles) of size $n\times n$ each, then the (i,j)-th block is N_{ij}.

We call M a one-tile matrix based on N if N appears as one of the tiles of M and the remaining tiles are zero.

Here is our first matrix-theoretic version of Theorem 1.

Theorem 6. *The space of invariants of degree nd on $\mathcal{X} := \mathrm{M}_n^{\oplus m}$ is spanned by those of the form*

$$\vec{X} = (X_1,\ldots,X_m) \mapsto \text{multi-det}(M_1,\ldots,M_{nd})$$

where each M_i is a one-tile $nd\times nd$ matrix based on one of the X_j's.

Proof. As explained in §2, it suffices to show that every multilinear invariant on nd copies of M_n can be expressed in the form stated in the theorem. Therefore we shall now assume that $m = nd$.

Let us now note that every function of the kind stated in the theorem is G-invariant. This follows from the fact that multi-det is G'-invariant for $G' = \mathrm{SL}(nd)\times\mathrm{SL}(nd)$ acting on M_{nd} (analogous to the G-action on M_n) and that the action of G on the multi-det is induced by the block-diagonal embedding of G in G'.

Now we proceed to prove that every multilinear invariant is a multi-determinant of suitable one-tile matrices. By (2) of Theorem 1, it suffices to consider the invariants of the kind $s_\sigma\otimes t_\tau$. For any such $\sigma,\tau\in S_{nd}$, and for $1\le i,j\le d$ consider the set

$$C_{ij} := \{\sigma((i-1)n+1),\ldots,\sigma(in)\} \ \cap\ \{\tau((j-1)n+1),\ldots,\tau(jn)\}.$$

We now assign to any $\vec{X} = (X_1,\ldots,X_{nd})\in\mathcal{X}$ a sequence of one-tile $nd\times nd$ matrices $(X_1^{\boxplus},\ldots,X_{nd}^{\boxplus})$ as follows. Each $X_k^{\boxplus}$ is based on X_k and the tiling position of X_k in $X_k^{\boxplus}$ is given by the unique pair (i,j) such that $k\in C_{ij}$.

We claim that $s_\sigma\otimes t_\tau$ satisfies the multilinear rule

$$(X_1,\ldots,X_{nd}) \mapsto \text{multi-det}(X_1^{\boxplus},\ldots,X_{nd}^{\boxplus}).$$

or equivalently, that the induced functional on $\mathrm{M}_n^{\otimes nd}$ determined by

$$X_1\otimes\ldots\otimes X_{nd} \mapsto \text{multi-det}(X_1^{\boxplus},\ldots,X_{nd}^{\boxplus})$$

equals $s_\sigma\otimes t_\tau$ via the usual identification of $V_i\otimes W_i$ with the i-th copy of M_n.

Let $E_{k,l}$ denote the elementary matrix having entry 1 at the spot (k,l) and zero elsewhere. Let us use $e_1,\ldots,e_n$ to denote the standard basis of $V = V_i$ and $f_1,\ldots,f_n$ for $W = W_j$. Recall that the identification $V\otimes W = \mathrm{M}_n$ sends $e_k\otimes f_l$ to $E_{k,l}$. Thus, to prove the claim above it suffices to show that for any two sequences of integers $(i_1,\ldots,i_{nd})$ and $(j_1,\ldots,j_{nd})$ coming from the set $\{1,\ldots n\}$, it holds that

$$(s_\sigma\otimes t_\tau)((e_{i_1}\otimes f_{j_1})\otimes\cdots\otimes(e_{i_{nd}}\otimes f_{j_{nd}})) = \text{multi-det}(E^{\boxplus}_{i_1,j_1},\ldots,E^{\boxplus}_{i_{nd},j_{nd}})$$

This is a straightforward verification. $\square$

For the rest of this section we provide yet another description of the matrix invariants.

Definition 7. A $d \times d$ matrix $L = (l_{ij})$ is said to be a magic square with row sum and column sum n, if each l_{ij} is an integer satisfying $0 \leq l_{ij} \leq n$ and the sum of the entries of each row and each column of the matrix equals n.

Definition 8. Let M be an $nd \times nd$ matrix of distinct indeterminates, and let $\{N_{ij}\}$ be the corresponding $n \times n$-tiling of M. Let L be a $d \times d$ magic square with row sum and column sum n. We say that a monomial expression f in the entries of M has grade L, if it has exactly l_{ij} indeterminates from the tile N_{ij}. In particular then, f necessarily has degree dn.

With notation as in the Definition 8, we set $\det(M)_L$ to be the sum of monomials in $\det(M)$ having grade L. Let us now choose an ordering of the tiles N_{ij}, the lexicographic ordering for instance. Via this ordering let us identify M_{nd} with $\mathrm{M}_n^{\oplus d^2}$. In particular, we may now identify the indeterminates in M with the usual coordinate functions on the space $\mathrm{M}_n^{\oplus d^2}$. Recall that $G = \mathrm{SL}(n) \times \mathrm{SL}(n)$ acts on $\mathrm{M}_n^{\oplus d^2}$ as before.

Lemma 9. *For any magic square L, the function $\det(M)_L$ is G-invariant. Moreover $\det(M) = \sum_L \det(M)_L$ where L ranges over all the magic squares.*

Proof. First note that $\det(M)$ is G-invariant. This follows from the same argument used for G-invariance of multi-det in the proof of Theorem 6 above. Next note that every monomial that appears in the expansion of $\det(M)$ has a grade corresponding to a suitable magic square. Thus $\det(M)$ is the sum of all the $\det(M)_L$'s. Finally, since the action of any $g \in G$ preserves the tiling, therefore g sends any monomial of grade L, to a sum of monomials with grade L. Thus $\det(M)_L$ is G-invariant. $\square$

For the sake of illustration, let us see that the invariant $\det(M)_L$ from Lemma 9 can also be realized as the multi-determinant of nd one-tile matrices. Let $L = (l_{ij})$. Let N_{ij} denote the $n \times n$ tile of M at the (i, j)-th position. For each (i, j), we consider l_{ij} number of copies of the one-tile $nd \times nd$ matrix $N_{ij}^{\boxplus}$ having N_{ij} as its unique nonzero tile at the position (i, j). Since $\sum_{ij} l_{ij} = nd$, we obtain, in all, nd number of one-tile matrices. The multi-determinant of these matrices is precisely $\det(M)_L$ multiplied by $\Pi_{ij}(l_{ij}!)$.

Now suppose we polarize $\det(M)_L$ (completely). In the above multi-det description, this amounts to distinguishing between the various copies of $N_{ij}^{\boxplus}$ by introducing a new set of indeterminates for each extra copy of N_{ij} used. Of course, only those (i, j) for which $l_{ij} \neq 0$ matter. Since $\sum_{ij} l_{ij} = nd$, the resulting expression involves nd tiles of indeterminates again and is moreover multilinear vis-a-vis each tile.

Looking at the proof of Theorem 6 we see that this multilinear invariant is in fact $s_\sigma \otimes t_\tau$, where for L one chooses l_{ij} to be the cardinality of the set C_{ij} constructed in the proof of the Theorem. It is easy to see that every magic square L arises from suitable σ, τ in this manner. To summarize, we have the following.

Proposition 10. *Every multilinear invariant on M_n of degree nd of the kind $s_\sigma \otimes t_\tau$ is a complete polarization of the invariant $\det(M)_L$ for a suitable $d \times d$ magic square L determined by σ, τ. Conversely, for every magic square L, the complete polarization of $\det(M)_L$ is of the kind $s_\sigma \otimes t_\tau$.*

Let us call the complete polarization of $\det(M)_L$ as $\mathrm{pd}(M)_L$.

We remark here that the invariants $s_\sigma \otimes t_\tau$, $\det(M)_L$ and $\mathrm{pd}(M)_L$ do not uniquely determine σ, τ or L. As remarked before, $s_\sigma \otimes t_\tau$ depends, upto a sign, only on the size n partitions of nd letters induced by σ and τ. For $\det(M)_L$ and $\mathrm{pd}(M)_L$, we note that permuting the rows and columns of L produces the same invariant upto renaming of variables and sign. Finally note that our identification of the space M_{nd} with $\mathrm{M}_n^{\oplus d^2}$ involved an ad-hoc choice of a suitable ordering.

Combining all that we have done so far, we get a succinct description of the generators of the degree nd piece of $R_{m,n}$.

Theorem 11. *Let $\{t_{ijs}\}$, $1 \leq i,j \leq d$, $1 \leq s \leq m$ be a set of md^2 indeterminates. For any $\vec{X} = (X_1, \ldots, X_m) \in \mathcal{X}$, let $N(\vec{X})$ be the $nd \times nd$ matrix whose (i,j)-th tile is the $n \times n$ matrix $\sum_{s=1}^{m} t_{ijs} X_s$. For any monomial $\mathfrak{m}$ that is a product of nd terms chosen from $\{t_{ijs}\}$, let $\mathfrak{m}(\vec{X})$ denote its coeffecient in $\det(N(\vec{X}))$. Then the set of functions of the type $\vec{X} \mapsto \mathfrak{m}(\vec{X})$ forms a generating set of the degree nd piece of the ring of invariants $R_{m,n}$.*

Proof. Let $\mathfrak{m} = t_{i_1 j_1 s_1} t_{i_2 j_2 s_2} \cdots t_{i_{nd} j_{nd} s_{nd}}$ (here the terms may actually be repeating) be a monomial that appears in $\det(N(\vec{X}))$. Corresponding to $\mathfrak{m}$, let l_{ij} be the cardinality of the set $\{k \mid i = i_k, j = j_k\}$. Then $L = (l_{ij})$ forms a magic square with row and column sum d and moreover every magic square arises this way from some monomial. It is easily verified that the function $\vec{X} \mapsto \mathfrak{m}(\vec{X})$ is nothing but the multilinear invariant $\mathrm{pd}(M)_L$ evaluated on $X_{s_1}, \ldots, X_{s_{nd}}$. Since $s_1, \ldots, s_{nd}$ can be chosen arbitrarily, the theorem follows from Theorem 1, part (2) and Proposition 10. $\square$

Remark 12. One can associate to the generator $\mathfrak{m}(\vec{X})$ from the previous theorem, a $d \times d$ matrix, also denoted by $\mathfrak{m}(\vec{X})$. The entries of $\mathfrak{m}(\vec{X})$, are m-tuples of non-negative integers $< l_{ij1}, l_{ij2}, \ldots, l_{ijm} >$, where l_{ijk} is the degree of t_{ijk} in $\mathfrak{m}$. Clearly for every i, $\Sigma_{jk} l_{ijk} = n$ and for every j, $\Sigma_{ik} l_{ijk} = n$. We call $\mathfrak{m}(\vec{X})$ a generalized magic square. We believe that this combinatorial data will be useful in understanding the $SL(m)$ structure of the invariants of degree nd. It is reasonable to conjecture that there is a set of generalized magic squares arising from linearly independent, spanning monomials $\mathfrak{m}$, and an A_m-type crystal structure on this set of magic squares. The underying crystal structure will then allow us to compute the desired multiplicity information.

Remark 13. The results above can be generalized to the case when $\mathrm{SL}(n) \times \mathrm{SL}(p)$ acts on m-tuples of $n \times p$ matrices. In this case invariants of degree d exist only when d is a multiple of the lcm of n, p. Invariants of degree d are again graded. This time the determinant of a $d \times d$ matrix, is tiled by $n \times p$-sized tiles. So multilinear invariants can be obtained using $d/n \times d/p$ rectangles, whose entries are non-negative integers such that the sum of the entries of each column is p and the sum of the entries of each row is n.

4. Birational invariant rings and invariants for conjugation action

We now focus our attention towards providing a smaller set of invariants of $\mathcal{X}$ of §2 that generate a subring which is birational to the full ring of invariants. These generators are chosen to be analogous to the trace monomials that appear in the Artin-Procesi description of invariants for the conjugation action. We also show that our description of invariants in §3 gives us a new though somewhat less efficient proof of the Artin-Procesi theorem.

In our setup the conjugation action is naturally induced via the obvious twisted diagonal embedding of $\mathrm{SL}(n)$ in G. Our calculations below exploit this embedding. Let us first set up some notation.

As before, let $\mathcal{X} = \mathcal{X}^{(m)} := \mathrm{M}_n^{\oplus m}$ be the space of m-tuples of $n \times n$ matrices together with a G-action as given in (2.1) above. Set $\mathcal{Y} := K \oplus \mathrm{M}_n^{\oplus(m-1)}$, i.e., $\mathcal{Y}$ consists of m-tuples whose first entry is an element of the field K and the remaining $m-1$ entries are from M_n. We write a typical element of $\mathcal{Y}$ as $(a, Y_2, \ldots, Y_m)$. We define an action of $H := \mathrm{SL}(n)$ on $\mathcal{Y}$ wherein any matrix $A \in H$ acts trivially on the first component K and via inner conjugation on the rest, i.e., $A \cdot Y_i = AY_iA^{-1}$. Clearly this action makes $\mathcal{Y}$ a rational H-module.

Let $\mathcal{Z}$ be the closed subset of $\mathcal{X}$ consisting of those m-tuples whose first entry is a scalar matrix. Thus $\mathcal{Z}$ can be naturally identified with $\mathcal{Y}$ and we shall also think of it as an H-module the same way. A typical element of $\mathcal{Z}$ may be written as $(\lambda \mathrm{I}, X_2, X_3, \cdots, X_m)$.

Let $\mathcal{X}^{\mathrm{o}}$, $\mathcal{Y}^{\mathrm{o}}$, $\mathcal{Z}^{\mathrm{o}}$ be open subsets of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ respectively, defined via non-vanishing of the first entry in case of $\mathcal{Y}$ and $\mathcal{Z}$ and non-vanishing of the determinant of the first matrix in case of $\mathcal{X}$. Each of these open subsets, being the complement of an invariant hypersurface, is affine and also a saturated subset, i.e., is a union of orbit closures too.

Consider the natural maps $\alpha \colon \mathcal{Z} \to \mathcal{X}$ and $\beta \colon \mathcal{X} \to \mathcal{Y}$ where α is the natural closed immersion and β is given by

$$(X_1, X_2, \ldots, X_m) \quad \mapsto \quad (\det(X_1), X_2X_1', \ldots, X_mX_1'),$$

where X_1' denotes the adjoint of the matrix X_1. Both α and β are compatible with the corresponding group actions via the homomorphisms $H \to G$ and $G \to H$ given respectively by

$$A \mapsto (A, (A^t)^{-1}), \qquad (A, B) \mapsto A.$$

In particular, the composite map $\beta\alpha$ is H-equivariant. Observe that $\beta^{-1}(\mathcal{Y}^{\mathrm{o}}) = \mathcal{X}^{\mathrm{o}}$ and $\alpha^{-1}(\mathcal{X}^{\mathrm{o}}) = \mathcal{Z}^{\mathrm{o}}$.

Consider the following commutative diagram of obvious natural induced maps.

$$(*) \qquad \begin{array}{ccccc}
\mathcal{Z}^{\mathrm{o}} & \longrightarrow & \mathcal{Z}^{\mathrm{o}}/H & \longrightarrow & \mathcal{Z}^{\mathrm{o}}/\!/H \\
{\scriptstyle\alpha^{\mathrm{o}}}\big\downarrow & & {\scriptstyle\alpha'}\big\downarrow & & {\scriptstyle\alpha''}\big\downarrow \\
\mathcal{X}^{\mathrm{o}} & \longrightarrow & \mathcal{X}^{\mathrm{o}}/G & \longrightarrow & \mathcal{X}^{\mathrm{o}}/\!/G \\
{\scriptstyle\beta^{\mathrm{o}}}\big\downarrow & & {\scriptstyle\beta'}\big\downarrow & & {\scriptstyle\beta''}\big\downarrow \\
\mathcal{Y}^{\mathrm{o}} & \longrightarrow & \mathcal{Y}^{\mathrm{o}}/H & \longrightarrow & \mathcal{Y}^{\mathrm{o}}/\!/H
\end{array}$$

The objects in the middle column are orbit spaces, i.e., they are merely sets whose elements represent orbits in the corresponding object of the leftmost column. The objects in the rightmost column are varieties given by the corresponding geometric quotients in the sense of Mumford. The horizontal arrows are all surjective maps of sets while the composite map across each row is a map of varieties. Also α'', β'' are maps of varieties.

Our main aim now is to show that β'' is an isomorphism. Before that we shall need a couple of lemmas.

Lemma 14.

(a) *The map $\beta'\colon \mathcal{X}^o/G \to \mathcal{Y}^o/H$ is a bijection.*
(b) *If $m \geq 3$, then the natural surjection $\mathcal{Y}^o/H \to \mathcal{Y}^o/\!/H$ is one-one over some open subset of the base.*
(c) *The composite map $\beta^o\alpha^o\colon \mathcal{Z}^o \to \mathcal{Y}^o$ is finite.*

Proof. (a). To prove that β' is surjective it suffices to show that β^o is surjective since the horizontal maps are all surjective. Surjectivity of β^o is obvious since for any $a \in K^*$, and any n-th root $a^{1/n}$ of a, we have

$$\beta\left(a^{1/n}\mathrm{I},\ a^{(1/n)-1}Y_2,\ \ldots,\ a^{(1/n)-1}Y_m\right) \quad = \quad (a, Y_2, \ldots, Y_m).$$

To prove injectivity, let $\vec{S} = (S_1, S_2, \ldots, S_m)$ and $\vec{T} = (T_1, T_2, \ldots, T_m)$ be elements of $\mathcal{X}^o$ mapping to the same element in $\mathcal{Y}^o/H$. Then $\det(S_1) = \det(T_1)$ and there exists $A \in H$ such that $AS_iS_1'A^{-1} = T_iT_1'$, for all $i \geq 2$. Hence the pair $(A,\ (S_1^{-1}A^{-1}T_1)^t) \in G$ sends $\vec{S}$ to $\vec{T}$.

(b). It suffices to show that the open subset $\mathcal{Y}^{o\prime}$ of $\mathcal{Y}^o$ consisting of points having closed H-orbits of maximum dimension is nonempty. Indeed, for any point $\vec{P}$ outside $\mathcal{Y}^{o\prime}$, the orbit closure of $\vec{P}$ cannot intersect $\mathcal{Y}^{o\prime}$ since the orbit dimension on the boundary is necessarily smaller. Therefore $\mathcal{Y}^{o\prime}$ is preserved under the semi-stable equivalence relation on $\mathcal{Y}^o$ (namely, $\vec{P} \sim \vec{Q}$ if the orbit closures of $\vec{P}$ and $\vec{Q}$ meet). Since $\mathcal{Y}^o/\!/H$ parameterizes the semi-stable equivalence classes of $\mathcal{Y}^o$, the image of $\mathcal{Y}^{o\prime}$ in $\mathcal{Y}^o/\!/H$ gives us the desired open set.

Now we show that the open subset of $\mathcal{Y}^o$ consisting of points having finite stabilizer in H is nonempty, i.e., we exhibit one point with finite stabilizer. Consider the m-tuple $\vec{Y} = (1, Y_2, Y_3, \ldots, Y_m)$ where $Y_4, \ldots, Y_m$ are allowed to be arbitrary and Y_2, Y_3 are defined as follows. For Y_2 we choose a diagonal matrix with distinct diagonal entries while for $Y_3 = (y_{ij})$ we set $y_{ij} = 1$ if $|i-j| = 1$ and $y_{ij} = 0$ otherwise. In other words, Y_3 has superdiagonal and subdiagonal entries 1 and rest 0. Let us now verify that $\vec{Y}$ has finite stabilizer. Indeed, any matrix A stabilizing Y_2 is necessarily a diagonal matrix with diagonal entries say $(a_1, \ldots, a_n)$. Since A stabilizes Y_3, we have $a_i = a_{i+1}$ and hence A is a scalar matrix given by the roots of unity.

At this point one can invoke a result of Popov ([12], Chapter 3, remarks after Lemma 3.9), to deduce that $\mathcal{Y}^{o\prime}$ is nonempty. We shall now give a direct proof of this by showing that for $\vec{Y}$ as chosen in the above paragraph, its H-orbit $O(\vec{Y})$ is in fact closed. Suppose

$$\vec{P} = (a, P_2, P_3, \ldots, P_m) \in \overline{O(\vec{Y})} \setminus O(\vec{Y}).$$

Then, by Hilbert-Mumford theory, (see [8]) there is a nonconstant one-parameter subgroup $\lambda\colon \mathbb{G}_\mathrm{m} \to H$ that drives $\vec{Y}$ to $\vec{P}$. Such a λ necessarily stabilizes $\vec{P}$. Since H acts via conjugation, the action preserves characteristic polynomials of the matrices involved and hence P_2 has eigenvalues same as those of Y_2. In particular, they are distinct. Since $\lambda(t)$ fixes P_2, it is a diagonal matrix for all $t \in \mathbb{G}_\mathrm{m}$. In particular, we may write the diagonal entries of $\lambda(t)$ as $(t^{a_1}, \ldots, t^{a_n})$ for suitable integers a_i satisfying $\sum_i a_i = 0$. Now $\lambda(t) \cdot Y_3$ is a matrix having entries $t^{a_i - a_{i+1}}$ and $t^{a_{i+1}-a_i}$. Hence for $\lim_{t\to 0} \lambda(t) \cdot Y_3$ to exist, it must hold that $a_i - a_{i+1} \geq 0$ and $a_i - a_{i+1} \leq 0$. Hence $a_i = a_{i+1}$. But this contradicts the fact that λ is nonconstant. (In effect, no $\mathbb{G}_\mathrm{m}$-orbit through $\vec{Y}$ has a boundary.)

(c). Both $\mathcal{Z}^{\mathrm{o}}$ and $\mathcal{Y}^{\mathrm{o}}$ are affine and $\beta^{\mathrm{o}}\alpha^{\mathrm{o}}$ is given by

$$(\lambda \mathrm{I}, X_2, X_3, \cdots, X_m) \quad \mapsto \quad (\lambda^n,\ \lambda^{n-1}X_2,\ \lambda^{n-1}X_3,\ \cdots,\ \lambda^{n-1}X_m).$$

Let $\Gamma(\mathcal{Z}) = K[l, (x_i)_{jk}]$, $2 \leq i \leq n$, $1 \leq j, k \leq n$ denote the coordinate ring of $\mathcal{Z}$ and likewise for $\mathcal{Y}$ we use $\Gamma(\mathcal{Y}) = K[d, (y_i)_{jk}]$. Then $\Gamma(\mathcal{Z}^{\mathrm{o}}) = K[l, 1/l, (x_i)_{jk}]$ and $\Gamma(\mathcal{Y}^{\mathrm{o}}) = K[d, 1/d, (y_i)_{jk}]$ and the natural map induced by $\beta^{\mathrm{o}}\alpha^{\mathrm{o}}$ sends

$$d \mapsto l^n, \qquad (y_i)_{jk} \mapsto l^{n-1}(x_i)_{jk}.$$

Then $(x_i)_{jk}$ satisfies the monic equation

$$(x_i)^n_{jk} = ((y_i)_{jk})^n (1/d)^{n-1}.$$

□

Lemma 15. *Let $\phi\colon R \to S$ be an integral extension of domains with R normal and suppose H is a group acting via homomorphisms on R, S in a way compatible with ϕ. Then the induced map of invariant rings $R^H \to S^H$ is also integral.*

Proof. Let $Q(R) \to Q(S)$ denote the corresponding algebraic extension of quotient fields. Let $s \in S^H$ and $\sigma \in H$. Since R is normal, the (unique) minimal monic polynomial of s over $Q(R)$ has coefficients in R, say

$$s^n + r_{n-1}s^{n-1} + \cdots + r_0 = 0.$$

Applying σ to this equation results in another such monic polynomial, whence each r_i is also invariant. □

An immediate consequence of part (c) of Lemma 14 and Lemma 15 is that the map β'' of the diagram (*) is a finite surjective map. For $m \geq 3$ we have a stronger statement.

Proposition 16. *For $m \geq 3$ the map β'' of the diagram (*) is an isomorphism.*

Proof. Let U be the open subset of $\mathcal{Y}^{\mathrm{o}}/\!/H$ as in (b) of Lemma 14. By (a) of Lemma 14, the inverse image of U in $\mathcal{X}^{\mathrm{o}}/G$ is bijective over U. Since the horizontal maps in (*) are surjective, hence β'' is bijective over U. As K has characteristic zero, we conclude that β'' is birational. By (c) of Lemma 14 and Lemma 15, $\beta''\alpha''$ is finite and hence β'' is also finite. Since $\mathcal{X}^{\mathrm{o}}/\!/G$ and $\mathcal{Y}^{\mathrm{o}}/\!/H$ are normal, the result follows. □

Set $\mathcal{W} := \mathrm{M}_n^{\oplus(m-1)}$ with H-action via conjugation in each component. Then $\mathcal{W}$ is a factor of $\mathcal{Z}$ and $\mathcal{Y}$ (also of $\mathcal{Z}^{\mathrm{o}}$ and $\mathcal{Y}^{\mathrm{o}}$), i.e., there are natural projections $\mathcal{Z} \to \mathcal{W}$ and $\mathcal{Y} \to \mathcal{W}$ which moreover admit section maps $\mathcal{W} \to \mathcal{Z}$ and $\mathcal{W} \to \mathcal{Y}$ that correspond to choosing for the first entry of an m-tuple, the identity element (the identity matrix I for $\mathcal{Z} \subset \mathcal{X}$ and $1 \in K$ for $\mathcal{Y}$). There results the following commutative diagram of obvious natural maps.

(**)
$$\begin{array}{ccccc}
\mathcal{W} & = & \mathcal{W} & = & \mathcal{W} \\
\downarrow & & & & \uparrow \\
\mathcal{Z} & \xrightarrow{\alpha} & \mathcal{X} & \xrightarrow{\beta} & \mathcal{Y}
\end{array}$$

Proposition 17. *Let d be the invariant function on $\mathcal{X}$ which assigns to any m-tuple, the determinant of the first term. Then, there is a natural integral extension of rings of invariants $\Gamma(\mathcal{W})^H[t, 1/t] \hookrightarrow \Gamma(\mathcal{X})^G[1/d] = \Gamma(\mathcal{X}^{\mathrm{o}})^G$ with $t \mapsto d$, t an indeterminate, which moreover is an isomorphism for $m \geq 3$.*

Proof. This follows from the above discussion and Proposition 16. $\square$

The isomorphism part of the above proposition can now be used to give a birational subring of the ring of invariants on $\mathcal{X}$. This requires invoking the Artin-Procesi theorem on invariants for the conjugation action. We postpone this application for now and instead first show how our description of the invariants in §3 can also be used to deduce (a complicated proof of) the Artin-Procesi theorem.

We shall use $R_{m,n}$ to denote the ring of invariants of $\mathcal{X} = \mathcal{X}^{(m)}$ and $P_{m,n}$ for the invariants under conjugation action.

Lemma 18. *Every invariant of $P_{m-1,n}$ is obtained from $R_{m,n}$ by specializing the first matrix X_1 to identity.*

Proof. From the diagram in (**) above we obtain maps $\mathcal{W} \to \mathcal{X} \to \mathcal{W}$ that compose to identity. Hence there are also natural maps $\Gamma(\mathcal{W})^H \to \Gamma(\mathcal{X})^G \to \Gamma(\mathcal{W})^H$ composing to identity. In particular, the last map is surjective. $\square$

Theorem 19. *The ring $P_{m,n}$ of invariants for simultaneous conjugation action is generated by the trace monomials of the form $\mathrm{Tr}(X_{i_1} X_{i_2} \cdots X_{i_k})$.*

Proof. The trace monomials in X_i's are clearly invariant. From Lemma 18 it follows that if we first rename X_i to X_{i+1} for $1 \le i \le n$, next take the elements of $R_{m+1,n}$, and finally set X_1 to identity, then we get all the elements of $P_{m,n}$.

Before proceeding further with the proof we first express multideterminant in terms of trace monomials.

Claim 20. *For any permutation $\pi \in S_p$ and $p \times p$ matrices $A_1, A_2, \dots, A_p$, set*

$$\mathrm{Tr}_\pi(A_1, A_2, \dots, A_p) \;:=\; \mathrm{Tr}(A_{i_1} A_{i_2} \cdots A_{i_k}) \;\cdots\; \mathrm{Tr}(A_{t_1} A_{t_2} \cdots A_{t_s}).$$

where $(i_1\, i_2\, \dots\, i_k)(j_1\, j_2\, \dots\, j_r) \;\cdots\; (t_1\, t_2\, \dots\, t_s)$ is the cycle decomposition of π. Then the multi-determinant of $p \times p$ matrices $A_1, A_2, \dots, A_p$ is obtained as

$$\mathrm{multidet}(A_1, A_2, \dots, A_p) = \sum_{\pi \in S_p} (\mathrm{sgn}\ \pi) \mathrm{Tr}_\pi(A_1, A_2, \dots, A_p).$$

Proof. Both sides are multilinear in the A_i's and hence we may consider them as functional on $\mathrm{M}_p^{\otimes p}$ in the obvious way. Via the usual identification $\mathrm{M}_p = V \otimes W$, if $e_1, \dots, e_p$ denote the standard basis of V and $f_1, \dots, f_p$ of W, then to prove the formula in the claim it suffices to show that both sides evaluate the basis elements as follows,

$$(e_{i_1} \otimes \cdots \otimes e_{i_p}) \otimes (f_{j_1} \otimes \cdots \otimes f_{j_p}) \mapsto \begin{cases} 0 & \text{if } \{i_1, \dots, i_p\} \text{ not all distinct,} \\ 0 & \text{if } \{j_1, \dots, j_p\} \text{ not all distinct;} \\ \mathrm{sgn}(\sigma \cdot \tau) & \text{if } i_k = \sigma(k) \text{ and } j_k = \tau(k) \text{ for } \sigma, \tau \in S_p. \end{cases}$$

Clearly the left hand side satisfies this property. For the right hand side, first note that Tr_π is the functional

$$\otimes_i (V_i \otimes W_i) \cong \otimes_i (V_i \otimes W_{\pi(i)}) \xrightarrow{\otimes_i (\mathsf{Tr}_{V_i, W_{\pi(i)}})} \otimes_i K = K$$

where $\mathsf{Tr}_{V,W}$ denotes the trace functional on $V \otimes W = \mathrm{M}_p$. (This follows easily by using the product map Pr described before Lemma 26 below.) In particular, Tr_π maps the basis element $(e_{i_1} \otimes \cdots \otimes e_{i_p}) \otimes (f_{j_1} \otimes \cdots \otimes f_{j_p})$ to 1 if $\pi(j_k) = i_k$ and 0 otherwise.

If $i_k = \sigma(k)$ and $j_k = \tau(k)$ for some $\sigma, \tau \in S_p$ then Tr_π is nonzero on the above basis element only for $\pi = \sigma \cdot \tau^{-1}$. Now assume that the i_k's are not all distinct. Then the subset $T \subset S_p$ of π's satisfying $\pi(j_k) = i_k$ is a coset of a certain subgroup in S_p which is in fact a direct product of nontrivial permutation subgroups of S_p and hence $\sum_{\pi \in T} \mathrm{sgn}(\pi) = 0$. □

Continuing with the proof of the theorem, by Theorem 6, $R_{m+1,n}$ is generated by the multi-determinant of one-tile matrices M_j based on the original matrices, the X_i's. The product of any collection of such one-tile matrices is either zero or again a one-tile matrix where the new tile is the corresponding product of the older ones. In view of the claim proved above, it follows that the mulitdeterminant of M_j's is the sum of products of traces of monomials in the underlying tiles of the M_j's and hence also the X_i's. Thus after putting X_1 to identity the invariants we arrive at are again polynomials in trace monomials on remaining X_i's. □

We now come to the main result of this section.

Theorem 21. *For any $1 \leq j \leq m$, the subring of $R_{m,n}$ generated by the trace monomials functions in the m matrices $X_1 X_j', \ldots, X_m X_j'$ is birational to $R_{m,n}$.*

Proof. For $m \geq 3$ the result follows from Proposition 17 and the Artin-Procesi theorem. For $m = 1$, we in fact have equality. It remains to consider the case $m = 2$. This is proved in Corollary 23 below. □

5. The $m = 2$ case and the $n = 2$ case.

In the two special cases stated here, we are able to give a smaller set of invariants that generate the full ring of invariants.

Let us start with $m = 2$.

Theorem 22. *Let $\mathcal{X} = \mathrm{M}_n^{\oplus 2}$ be the space of pairs of $n \times n$ matrices with a G-action as before. For any i, let p_i be the polynomial function on $\mathcal{X}$ that assigns to any pair $(X_1, X_2) \in \mathcal{X}$, the coefficient of $t_1^{n-i} t_2^i$ in $\det(t_1 X_1 + t_2 X_2)$. Then $\{p_0, \ldots, p_n\}$ are algebraically independent over K and moreover generate the ring of invariant functions on $\mathcal{X}$.*

Proof. We note that $p_0 = \det(X_1)$ while $p_n = \det(X_2)$. Clearly the p_i's are invariant functions. Their algebraic independence follows easily by looking at the bidegree of these polynomials in the variables associated to X_1 and X_2.

Consider the G-invariant map $i \colon \mathcal{X} \to \mathbb{A}_K^{n+1}$ given by

$$(X_1, X_2) \ \mapsto \ (p_0(X_1, X_2), \ \ldots \ , \ p_n(X_1, X_2))$$

where the action of G on the affine $(n+1)$-space $\mathbb{A}_K^{n+1}$ is trivial. To complete the proof of the theorem it suffices to give a regular section $s \colon \mathbb{A}_K^{n+1} \to \mathcal{X}$ of i whose image $Z \subset \mathcal{X}$ (which is necessarily a closed subset isomorphic to $\mathbb{A}_K^{n+1}$ via i) is such that its G-span is dense in $\mathcal{X}$. Indeed, for any such s and Z, if f is an invariant function on $\mathcal{X}$, then fsi is an invariant function on $\mathcal{X}$ that agrees with f on Z and hence must equal f on all of $\mathcal{X}$. Since fs is a regular function on $\mathbb{A}_K^{n+1}$, therefore $f = fsi$ lies in the subring generated by the p_i's.

The desired section s is the one which sends the point $u = (u_0, u_1, \ldots, u_n)$ to the following pair of matrices.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & u_0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{n+1}u_n \\ 1 & 0 & \cdots & 0 & (-1)^n u_{n-1} \\ \vdots & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & 1 & 0 & -u_2 \\ 0 & 0 & 0 & 1 & u_1 \end{pmatrix}$$

Clearly s is regular. The verification $is =$ identity is straightforward. □

Corollary 23. *For $j = 1, 2$ the subring generated by the trace monomial functions on the two matrices $X_1X'_j$, $X_2X'_j$ is birational to $R_{2,n}$.*

Proof. Let us take the case $j = 1$. Then the trace monomial functions are polynomial expressions in $\det(X_1)$ and the coeffecients of the characteristic polynomial of $X_2X'_1$ (since K has characteristic is 0). Now the result follows easily from the above theorem by using

$$\det(t_1X_1 + t_2X_2)\det(X'_1) = \det(t_1X_1X'_1 + t_2X_2X'_1).$$

□

Let us now move on to the case $n = 2$. In this case we show that the birational subring of invariants given in Theorem 21 can in fact be used to obtain the full ring.

Theorem 24. *Let $\mathcal{X} = \mathrm{M}_2^{\oplus m}$ be the space of m-tuples of 2×2 matrices and let $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$ act on $\mathcal{X}$ as before. Then a generating set for the ring of invariants of $\mathcal{X}$ is given by functions that assign to any m-tuple $(X_1, \ldots, X_m) \in \mathcal{X}$, the trace of a monomial in the m^2 matrices $Y_{ij} := X_iX'_j$.*

Let us use the notation as in §2. In view of Theorem 1, it suffices to prove the following result.

Proposition 25. *Via the isomorphism $\mathcal{X} := \mathrm{M}_2^{\oplus m} \cong \otimes_{i=1}^m (V_i \otimes W_i)$, any invariant of of degree m of the kind $s_\sigma \otimes t_\tau$ evaluates an m-tuple $\vec{X} = (X_1, \ldots, X_m)$ to a product of trace monomials in the m^2 matrices $Y_{ij} := X_iX'_j$.*

Before proceeding with a proof of this proposition, we need a few elementary remarks. Let $V = K^2 = W$. In what follows, we make the identification $V \otimes W = \mathrm{M}_2$ as before. Let $\mathsf{Tr}_{V,W}$ denote the functional on $V \otimes W$ that sends a matrix to its trace and let $\mathsf{Adj}_{V,W} : V \otimes W \to V \otimes W$ denote the linear involution that sends a matrix X to its adjoint X'. Finally, let $\mathsf{Pr}_{V_1,W_1,V_2,W_2} : \mathrm{M}_2 \otimes \mathrm{M}_2 \to \mathrm{M}_2$ denote the natural map

$$(V_1 \otimes W_1) \otimes (V_2 \otimes W_2) \cong V_1 \otimes (V_2 \otimes W_1) \otimes W_2 \xrightarrow{1 \otimes \mathsf{Tr}_{V_2,W_1} \otimes 1} V_1 \otimes W_2$$

which in fact sends $X \otimes Y \in \mathrm{M}_2 \otimes \mathrm{M}_2$ to the product matrix XY. The following lemma follows easily from direct calculations.

Lemma 26. *Let $\phi \colon V \to W$ be the isomorphism sending $e_1 \mapsto -e_2$, $e_2 \to e_1$.*

(1) *The composition of the following maps equals $\mathsf{Adj}_{V,W}$.*

$$V \otimes W \xrightarrow{\phi \otimes 1} W \otimes W \xrightarrow{1 \otimes \phi^{-1}} W \otimes V \xrightarrow{w \otimes v \mapsto v \otimes w} V \otimes W$$

(2) *The following diagram commutes.*

$$\begin{tikzcd}
V \otimes V \arrow[r, "1\otimes\phi"] \arrow[d, "\text{canonical}"'] & V \otimes W \arrow[r, "\phi\otimes 1"] \arrow[d, "\mathsf{Tr}_{V,W}"] & W \otimes W \arrow[d, "\text{canonical}"] \\
\wedge^2 V \arrow[r] & K & \wedge^2 W \arrow[l]
\end{tikzcd}$$

Proof of Proposition 25. We proceed by induction on m. By Theorem 1, m is an even number. By permuting the m entries of $\mathrm{M}_2^{\oplus m}$ if necessary, we may assume without loss of generality that σ is the identity permutation, $\tau(1) = 1$ and $\tau(2)$ is either 2 or 3. Let $\vec{X} = (X_1, \ldots, X_m) \in \mathrm{M}_2^{\oplus m}$. If $\tau(2) = 2$, then it is clear that $s_\sigma \otimes t_\tau$ is a product of multi-det$(X_1, X_2) = \mathrm{Trace}(X_1' X_2)$ and an invariant $s_{\sigma'} \otimes t_{\tau'}$ on the remaining $m-2$ entries $(X_3, \ldots, X_m)$ for suitable $\sigma', \tau' \in S_{m-2}$. By induction, $s_{\sigma'} \otimes t_{\tau'}$ is a product of desired trace monomials, whence so is $s_\sigma \otimes t_\tau$.

Let us now assume that $\tau(2) = 3$. Recall that $V_i \otimes W_i$ denotes the i-th component of $\mathrm{M}_2^{\oplus m}$. We now apply the preceding lemma. First we use ϕ to replace V_1 by W_1 and W_1 by V_1. By part (1) of the lemma this amounts to replacing the matrix X_1 by X_1'. Moreover, by part (2) of the lemma, the determinant of V_1 with V_2 is transformed into Tr_{V_2,W_1} while the determinant of W_1 with W_3 is transformed to Tr_{V_1,W_3}. Using the description of the product map Pr, it now follows that $(s_\sigma \otimes t_\tau)(\vec{X}) = (s_{\sigma'} \otimes t_{\tau'})(\vec{Z})$ where $\vec{Z} = (X_3 X_1' X_2,\, X_4, \ldots, X_m) \in \mathrm{M}_2^{\oplus m-2}$ and σ' is the identity permutation on $(3, \ldots, m)$ while τ' is the restriction of τ to $(2, 4, 5, \ldots, m)$. By induction $s_{\sigma'} \otimes t_{\tau'}$ is a trace monomial of the desired kind on the entries of $\vec{Z}$ and it is now straightforward to check that $(s_{\sigma'} \otimes t_{\tau'})(\vec{Z})$ is a trace monomial of the desired kind on the X_i's.

□

6. The Null cone

When a group acts linearly on an affine space, the null cone associated to this action is the common zero locus of the invariant polynomials. In the context of constructing projective quotients for the group action, the points of the null cone are also called unstable points (for reductive groups). We now give a simple characterization of the null cone in our setup of $G = \mathrm{SL}(n) \times \mathrm{SL}(n)$ acting on $\mathcal{X} = \mathrm{M}_n^{\oplus m}$. Here we allow the field K to have arbitrary characteristic. As mentioned in the remark at the end of our introduction, this result already appears in [3]. The proof is also similar, but we avoid the language of the max-flow min-cut theorem used in [3].

Proposition 27. *Let $\mathcal{X} = \mathrm{M}_n^{\oplus m}$ be the space of m-tuples of $n \times n$-matrices with G action as before. Then an m-tuple $(X_1, X_2, \ldots, X_m) \in \mathcal{X}$ lies in the null cone $\mathcal{N}$ if and only if there there exists an integer r, $1 \le r \le n$ and subspaces W_1, W_2 of K^n having dimensions $r, r-1$ respectively, such that each X_i maps W_1 inside W_2.*

Proof. Let us first show that every point in the null cone has this property. Suppose $\vec{P} = (P_1, P_2, \cdots, P_m) \in \mathcal{N}$. Then by Hilbert-Mumford theory, there is a one-parameter subgroup (1-ps, in short) $\lambda\colon \mathbb{G}_\mathrm{m} \to G$ which drives $\vec{P}$ to the origin $\vec{0}$ of $\mathcal{X}$, i.e., as $t \in \mathbb{G}_\mathrm{m} \subset \mathbb{A}_K^1$ approaches 0, it holds that $\lambda(t) \cdot \vec{P}$ approaches $\vec{0}$.

Composing λ with the two natural projections $\pi_i\colon G \to \mathrm{SL}(n)$ results in two 1-ps's $\lambda_i\colon \mathbb{G}_\mathrm{m} \to \mathrm{SL}(n)$. Since the image of each λ_i is a torus, there exist $S, T \in \mathrm{SL}(n)$

such that for all t, $S\lambda_1(t)S^{-1}$ and $T\lambda_2(t)T^{-1}$ are diagonal subgroups of $\mathrm{SL}(n)$. For any sequence $\mathfrak{c} = (c_1, \ldots, c_n)$ of integers satisfying $\sum_i c_i = 0$ let us denote the corresponding diagonal subgroup of $\mathrm{SL}(n)$ with diagonal entries $(t^{c_1}, \ldots, t^{c_n})$ as $\mathrm{diag}(t^{c_1}, \ldots, t^{c_n})$ or simply $\mathrm{D}(\mathfrak{c})$. Thus we may write

$$S\lambda_1(t)S^{-1} = \mathrm{diag}(t^{a_1}, \ldots, t^{a_n}) = \mathrm{D}(\mathfrak{a}), \qquad T\lambda_2(t)T^{-1} = \mathrm{diag}(t^{b_1}, \ldots, t^{b_n}) = \mathrm{D}(\mathfrak{b}),$$

for suitable sequences of integers $\mathfrak{a}, \mathfrak{b}$ such that $\sum_i a_i = 0 = \sum_i b_i$.

Consider the point $\vec{Q} := (S, T) \cdot \vec{P}$ lying in the orbit of $\vec{P}$. In view of the definition of the G-action, we also write $\vec{Q} = S\vec{P}T^{\mathrm{t}}$ for convenience. We claim that the 1-ps $(\mathrm{D}(\mathfrak{a}), \mathrm{D}(\mathfrak{b}))$ of G drives $\vec{Q}$ to $\vec{0}$. Since $\vec{0} = \lim_{t\to 0} \lambda(t) \cdot \vec{P} = \lim_{t\to 0} \lambda_1(t)\vec{P}\lambda_2(t)^{\mathrm{t}}$, we deduce that upon applying $\lim_{t\to 0}$, the following expressions

$$(S\lambda_1(t)S^{-1},\ T\lambda_2(t)T^{-1}) \cdot \vec{Q} \ = \ (S\lambda_1(t),\ T\lambda_2(t)) \cdot \vec{P} \ = \ S\lambda_1(t)\vec{P}\lambda_2(t)^{\mathrm{t}}T^{\mathrm{t}}$$

evaluate to $\vec{0}$, whence the claim follows. To prove the proposition for $\vec{P}$, it suffices to find corresponding subspaces W_1', W_2' for $\vec{Q}$, for then $T^{\mathrm{t}}(W_1'), S^{-1}(W_2')$ will work for $\vec{P}$.

To simplify notation, we now work with only one of the matrices in $\vec{Q}$, say the first one Q_1, which we call Q. As will be obvious from the argument below, this simplification does not affect the final conclusion. Without loss of generality we may assume that $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are decreasing sequences. Multiplication by $\mathrm{D}(\mathfrak{a})$ on the left and $\mathrm{D}(\mathfrak{b})$ on the right sends the $(i\,j)$-th entry q_{ij} of Q to $t^{a_i+b_j}q_{ij}$. Since Q is being driven to zero, we must have $q_{ij} = 0$ whenever $a_i + b_j \leq 0$. We claim that there exists an r such that $a_r + b_{n+1-r} \leq 0$. Since $0 = \sum_i a_i + \sum_j b_j = \sum_i (a_i + b_{n+1-i})$, the claim follows. Hence $q_{ij} = 0$ for $i \geq r, j \geq n + 1 - r$ as a_i's and b_j's are decreasing. Thus Q maps the subspace spanned by the basis vectors $\{e_{n-r+1}, \ldots, e_n\}$ inside the subspace spanned by $\{e_1, e_2, \ldots, e_{r-1}\}$. (For $r = 1$, the latter is chosen as (0)).

The converse works out in a similar manner. Suppose $\vec{P}$ maps W_1 inside W_2. Choose $S, T \in \mathrm{SL}(n)$ such that T^{t} maps the span W_1' of $\{e_{n+1-r}, \ldots, e_n\}$ to W_1 and S^{-1} maps the span W_2' of $\{e_1, e_2, \ldots, e_{r-1}\}$ to W_2. Then $\vec{Q} := (S, T) \cdot \vec{P}$ maps W_1' inside W_2'. Hence for every matrix Q_k of $\vec{Q}$, its $(i,\, j)$-th entry is zero for $i \geq r,\ j \geq n + 1 - r$. We now show that under these conditions all the Q_k's can be simultaneously driven to zero by a suitable 1-ps in G, or equivalently, a pair of diagonal 1-ps's, say $\mathrm{D}(\mathfrak{a}), \mathrm{D}(\mathfrak{b})$ in $\mathrm{SL}(n)$.

For $r = 1$ we choose $\mathfrak{a} = (0, \ldots, 0)$ and $\mathfrak{b} = (1, 1, \ldots, 1, -n + 1)$. Similary, for the other extreme case $r = n$, we choose $\mathfrak{a} = (1, 1, \ldots, 1, -n+1)$ and $\mathfrak{b} = (0, \ldots, 0)$. For the rest, first choose a rational number γ such that $1 < \gamma < (r/(r-1))((n-r+1)/(n-r))$. Set

$$\alpha := (n - r + 1)/(r - 1), \quad \beta := -1, \quad \delta := -\gamma(n - r)/r.$$

Note that $\alpha + \delta > 0$ and $\beta + \gamma > 0$. Choose an integer N such that $N\alpha, N\beta, N\gamma, N\delta$ are all (nonzero) integers. Set

$$\mathfrak{a} = (\overbrace{N\alpha, \ldots, N\alpha}^{r-1 \text{ times}}, \overbrace{N\beta, \ldots, N\beta}^{n-r+1 \text{ times}}), \qquad \mathfrak{b} = (\overbrace{N\gamma, \ldots, N\gamma}^{n-r \text{ times}}, \overbrace{N\delta, \ldots, N\delta}^{r \text{ times}}).$$

It is a straightforward verification that $(\mathrm{D}(\mathfrak{a}), \mathrm{D}(\mathfrak{b}))$ drives each Q_k to zero.

Since the orbit closure of $\vec{Q}$ contains $\vec{0}$, every invariant function vanishes at $\vec{Q}$, i.e., it is in the null cone. Thus $\vec{P}$ is also in the null cone. □

The points arising from the two extreme cases $r = 1$ and $r = n$ of the proposition can also be interpreted as follows. An m-tuple $\vec{P}$ satisfies the case $r = n$, iff the $n \times mn$ matrix obtained by row-wise concatenation of P_i's has row rank less than n. Such points form the null cone for the corresponding Grassmannian of $n \times mn$ matrices under left multiplication by $\mathrm{SL}(n)$. Likewise, $\vec{P}$ satisfies $r = 1$ iff the $mn \times n$ matrix obtained by column-wise concatenation of P_i's has column rank less than n. Such points give the null cone for the corresponding Grassmannian of $mn \times n$ matrices under right multiplication by $\mathrm{SL}(n)$.

7. Conclusions

A lot of questions remain to be answered—the most important one being the $\mathrm{SL}(m)$-module decomposition of the space of invariants. This is ongoing work. Another interesting question is to determine upper bounds on the degree of the generators for the ring $R_{m,n}$. A further natural question to consider is that of studying the algebro-geometric properties of the quotient variety.

Most of our results have been in characteristic zero. One could ask the same question in characteristic p, namely what are the invariants in characteristic p for the simultaneous left-right action of $\mathrm{SL}(n) \times \mathrm{SL}(n)$ on m-tuples of $n \times n$ matrices ? It is tempting to conjecture that our description of invariants carries over to characteristic p. We know from Donkin's work, [5], that for the simultaneous conjugation action on m-tuples of matrices, the coeffecients of the characteristic polynomials of monomials in the matrices generate the ring of invariants in all characteristic. The trace monomials only work in characteristic zero. Thus our motivation partly has also been of providing trace-free descriptions of the invariants in our setup. As in Donkin's work this may lead to some deeper connections with Kazhdan Lusztig theory or a suitable analogue thereof.

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